SOME GEOMETRIC RESULTS ARISING FROM THE BOREL FIXED PROPERTY

JEAMAN AHN*, JUAN C. MIGLIORE**

Abstract. In this paper, we will give some geometric results using generic initial ideals for the degree reverse lex order. The first application is to the regularity of a Cohen-Macaulay algebra, and we improve a well-known bound. The main goal of the paper, though, is to improve on results of Bigatti, Geramita and Migliore concerning geometric consequences of maximal growth of the Hilbert function of the Artinian reduction of a set of points. When the points have the Uniform Position Property, the consequences they gave are even more striking. Here we weaken the growth condition, assuming only that the values of the Hilbert function of the Artinian reduction are equal in two consecutive degrees, and that the first of these degrees is greater than the second reduction number of the points. We continue to get nice geometric consequences even from this weaker assumption. However, we have surprising examples to show that imposing the Uniform Position Property on the points does not give the striking consequences that one might expect. This leads to a better understanding of the Hilbert function, and the ideal itself, of a set of points with the Uniform Position Property, which is an important open question. In the last section we describe the role played by the Weak Lefschetz Property (WLP) in this theory, and we show that the general hyperplane section of a smooth curve may not have WLP.

1. Introduction

It has been shown in work of Conca, Fløystad, Green and many others that generic initial ideals contain a tremendous amount of geometric information about the subschemes from which they are obtained. This paper continues in this vein, applying the theory of generic initial ideals to the study of geometric consequences that arise when the Hilbert function has certain growth properties (but possibly short of maximal growth). Along the way, we also obtain results about the regularity of a Cohen-Macaulay ring. Our main application is to the case of (reduced) points in projective space, especially those having the Uniform Position Property (UPP) (i.e. the property that any two subsets of the same cardinality have the same Hilbert function). Here what is *not* true turns out to be as striking as what *is* true, and we give some new insight into the behavior of points with UPP.

We begin in section 2 with a review of the basic facts about generic initial ideals. In the process, we make some new observations. Our main objects of interest are two natural invariants of a Borel fixed monomial ideal (see the definition of $D(A)$

* The first author expresses his gratitude to the University of Notre Dame Mathematics Department and to Juan Migliore for their hospitality. He also thanks BK21 and Younghyun Cho, his advisor; without their support, this paper would not have been written.
** Part of the work for this paper was done while the second author was sponsored by the National Security Agency under Grant Number MDA904-03-1-0071.
2000 Mathematics Subject Classification: 13D40, 14M05, 13C05, 13P10.

and $M(A)$ on page 6). We show how these invariants are related to many other important invariants such as the depth, codimension, etc. We also obtain results about the vanishing and non-vanishing of cohomology of the ideal sheaf of a closed subscheme of projective space, and a criterion about the Cohen-Macaulayness of the coordinate ring of a homogeneous ideal. In this section we also recall the definition of an invariant which will be central to the rest of the paper, namely the *s-reduction number* of R/I, which was introduced by Hoa and Trung [19]. This has several equivalent formulations (Lemma 2.15 gives some of them), but the most convenient for us is that $r_s(R/I) = \max\{k \mid H(R/(I+J),k) \neq 0\}$, where J is the ideal generated by s generally chosen linear forms.

The computation of the second reduction number, for a zero-dimensional scheme Z, is closely connected to the Weak Lefschetz Property (WLP) for the Artinian reduction of R/I_Z. WLP may be defined as follows: If $J = (L_1, L_2)$ is an ideal of generally chosen linear forms, then WLP is equivalent to the property that

$$H(R/(I_Z+J),t) = \max\{\Delta^2(R/I_Z,t), 0\}.$$

In particular, if WLP holds for the Artinian reduction of R/I_Z then

$$r_2(R/I_Z) = \max\{t | \Delta H(R/I_Z,t) > \Delta H(R/I_Z,t-1)\}.$$

It is very natural to ask whether a set of points with UPP automatically has WLP, since this property is the "expected" one and UPP is somehow a "general" property. We remark that we were somewhat surprised by Example 6.9, which gives a set of points with UPP that does not have WLP. Furthermore, since a general hyperplane or hypersurface section of a smooth curve has UPP, it is also very natural to ask whether the same is true for WLP. We were equally surprised to see that the same example can be modified to answer both of these questions in the negative; this is shown in Example 6.10.

In section 3 we apply the ideas of section 2 to give a bound on the regularity of an arithmetically Cohen-Macaulay projective subscheme. We generalize the well known fact (cf. [11]) that

$$\operatorname{reg}(I_X) \leq \deg(X) - \operatorname{codim}(X) + 1$$

by proving that if X is an arithmetically Cohen-Macaulay subscheme of $\mathbb{P}^{n-1}$ with $\operatorname{codim}(X) = e$ then

$$\operatorname{reg}(I_X) \leq \deg(X) - \binom{\alpha - 1 + e}{\alpha - 1} + \alpha,$$

where $\alpha = \alpha(I_X)$ is the initial degree of I_X (see Theorem 3.3). This was also proved by Nagel (who in fact gave a more general statement, but with a very different proof) in [22].

In the paper [4], the authors studied some geometric consequences that arise from maximal growth of the first difference, ΔH, of the Hilbert function of some projective subscheme. This has seen the most interest and applications when the subscheme is a finite set of (reduced) points, Z (so ΔH is the h-vector, i.e. the Hilbert function of the Artinian reduction), and here the most interesting case (from our point of view) comes when the maximal growth is given by $\Delta H(R/Z,d) = \Delta H(R/I_Z,d+1) = s$ (say). In this case, to be maximal growth means that necessarily $d \geq s$. When this equality on ΔH occurs for $d \geq s$, it was shown in [4] that the component $(I_Z)_d$ defines a *reduced* curve, V, of degree s, and that in fact $\operatorname{reg}(I_V) \leq d$. In this context it is not necessarily true that V is unmixed. Let C

be the pure one-dimensional part of V. When, in addition, we assume that Z has UPP, it was also shown that $Z \subset C$, $(I_Z)_d = (I_C)_d$, and that C is reduced and irreducible. This has strong implications for the possible Hilbert functions of sets of points with UPP, although it is still far from a classification. For example, it was not proved in [4], but we remark in Proposition 5.3 that these conditions imply that once $\Delta H(R/I_Z, t) < \Delta H(R/I_Z, t-1)$ for some $t > d+1$, then the Hilbert function is strictly decreasing from then on.

Section 4 begins the main geometric task in this paper, which is weaken the assumption $d \leq s$ of [4]. Our results are given in terms of the second reduction number, $r_2(R/I_Z)$. As mentioned above, if J is the ideal generated by two general linear forms, then $r_2(R/I_Z)$ is the maximum degree in which $R/(I_Z+J)$ is non-zero (we do not necessarily have WLP here). We note in Remark 4.4 that $d > r_2(R/I_Z)$ really is a weaker hypothesis than $d \geq s$. We also note in Remark 4.5 that in general it is much weaker, and that the hypothesis cannot be weakened further.

Our first main result is that if

$$\Delta H(R/Z, d) = \Delta H(R/I_Z, d+1) = s \text{ for some } d > r_2(R/I_Z) \tag{1.1}$$

then again (as in [4]) $(I_Z)_d$ is the degree d component of the saturated ideal of some curve V (not necessarily unmixed) of degree s, and $\operatorname{reg}(I_V) \leq d$. Easy examples show that this cannot be extended to $d \leq r_2(R/I_Z)$.

Enboldened by this result, in section 5 we sought to prove results analogous to those of [4] for the case where Z has, in addition, UPP. We found, on the contrary, some very surprising examples, and in doing so we have gained some new insight (albeit negative) into the behavior of the Hilbert function of points with UPP. It is not as well-behaved as one might think. Luckily, we do retain the fact that if (1.1) holds for a set, Z, with UPP then $Z \subset C$ (Theorem 5.2), where C is the unmixed part of V. That is, $V = C$ is unmixed.

It is known that if Z is a set of points with UPP, a general element of smallest degree of I_Z is reduced and irreducible. For points in $\mathbb{P}^3$, for instance, this means that a general surface of least degree containing Z is reduced and irreducible. In particular, when this surface is unique, it is reduced and irreducible. We expected that if Z is a finite set of points with UPP, then in the least degree component of I_Z where the base locus is a curve, C, we would analogously get that C is reduced and irreducible; it was shown in [4] that this is true when $d \geq s$. On the contrary, however, Example 5.7 shows that if we assume only condition (1.1) and UPP, we do get an unmixed curve C containing Z (as noted above), but C can fail to be either reduced or irreducible. We also expected that even in this situation it would continue to be true that if Z has UPP and satisfies (1.1), then once $\Delta H(R/I_Z, t) < \Delta H(R/I_Z, t-1)$ for some $t > d+1$, then the first difference of the Hilbert function would be strictly decreasing from then on. But on the contrary, Example 5.7 also shows that this is false. (As mentioned above, though, Proposition 5.3 gives conditions where the first difference of the Hilbert function *is* strictly decreasing.) One might think that somehow Example 5.7 was an "accident," in the sense that in Example 5.7, while the curve C (defined by the degree d component of Z) does fail to be either reduced or irreducible, still there is a reduced and irreducible component of C that contains all of the points. Perhaps at least this is always true for points with UPP. But in fact, Example 5.9 removes even this hope. This is an example of a set of points with UPP, satisfying condition (1.1), for which the curve defined by the component of degree d is reduced and

consists of two "identical" components, each of which contains exactly half of the points.

Our results in section 6 were motivated both by the desire to obtain some results in the range $d \leq r_2(R/I_Z)$, and also by the desire to understand the role of WLP in this theory beyond simply being a useful tool for computing $r_2(R/I_Z)$. Now it turns out that the *second* difference of the Hilbert function and $r_3(R/I_Z)$ are important. One has to be more careful here, since reducing by a second linear form does not necessarily have the same Hilbert function as the second difference of the original Hilbert function; this does hold when we have WLP, and we use this fact. For instance, we show in Theorem 6.7 that if Z is a zero-dimensional subscheme of $\mathbb{P}^{n-1}$, $n > 3$, with WLP and if

$$\Delta^2 H(R/I_Z, d) = \Delta^2(R/I_Z, d+1) = s$$

for $r_2(R/I_Z) > d > r_3(R/I_Z)$, then $\langle (I_Z)_{\leq d} \rangle$ is a saturated ideal defining a two-dimensional scheme of degree s in $\mathbb{P}^{n-1}$, and it is d-regular. We also show in Corollary 6.12 that if Z has both UPP and WLP, and if I_Z has generators in degrees $d_1 \leq d_2 \leq \dots$, then

$$\Delta^2 H(R/I_Z, d) > \Delta^2 H(R/I_Z, d+1)$$

for $d_2 \leq d < r_2(R/I_Z)$. As mentioned above, we also show (Example 6.9) that UPP does not necessarily imply WLP, and in Example 6.10 we modify this example to show that the general hyperplane or hypersurface section of a smooth curve does not necessarily have WLP (although it does necessarily have WLP if the curve is in $\mathbb{P}^3$).

Finally, we would like to comment on this invariant $r_2(R/I_Z)$. It is seen in this paper to be a very natural invariant, and when WLP holds it is easy to compute. WLP has been shown in other papers to be a very commonly occuring property, as has UPP, and in section 6 we have shown that it plays an integral role in our study of Hilbert functions. However, for an arbitrary set of points it is not always easy to determine whether WLP holds, nor is it always easy to compute $r_2(R/I_Z)$. This makes our results less easy to apply than, for instance, the corresponding results of [4] where it is assumed that $d \geq s$. However, it is not hard to see that $r_2(R/I_Z)$ can be much smaller than s. Our methods thus allow us to give results in a (possibly large) range in which the methods of [4] absolutely do not apply. This results in new information about the Hilbert function of a set of points with UPP. We also show that very interesting subtle differences arise between what happens in this range and what happens in the range $d \geq s$, leading to interesting examples of unusual behavior in the Hilbert function of a set of points with UPP. We believe that our methods will lead to further progress on these questions.

We are grateful to Uwe Nagel for asking us the question that led to Example 6.10.

2. Preliminaries

In this section, we survey some definitions, notation and some preliminary facts for generic initial ideals. Let $R = K[x_1, \dots, x_n]$ be the polynomial ring over a field K of characteristic 0. For any $g = (g_{ij}) \in \mathrm{GL}_n(R_1^\vee)$, we define an action on R which induces an k-algebra isomorphism for any homogeneous form $f \in R$ by

$$f(x_1, \dots, x_n) \mapsto f(g(x_1), \dots, g(x_n)),$$

where $g(x_i) = \sum_{j=1}^{n} g_{ij} x_j$. A monomial ideal I is said to be *Borel fixed* if

$$g(I) = I$$

for every upper triangular matrix $g \in \mathrm{GL}_n(R_1^\vee)$. A Borel fixed monomial ideal has a nice combinatorial property, namely that of being *strongly stable*: If $x_i m \in I$ for some monomial $m \in I$, then $x_j m \in I$ for all $j < i$.

For any monomial term order τ, the initial ideal of a homogeneous ideal $I \subset R$ depends on the choice of variables and basis made. By allowing a generic change of basis and coordinates, we may eliminate this dependence.

Theorem 2.1. ([14], [2], [10]) *For any monomial term order τ and any homogeneous ideal I, there is a Zariski open subset $U \subset \mathrm{GL}_n(R_1^\vee)$ such that $\mathrm{in}_\tau(g(I))$ is constant and Borel fixed for $g \in U$. We will call $\mathrm{in}_\tau(g(I))$ the generic initial ideal of I and denote it $\mathrm{Gin}_\tau(I)$.*

Note that a homogeneous ideal I is *saturated* if

$$(I : (x_1, \ldots, x_n)) = I.$$

The *saturation* of I is

$$I^{sat} = \bigcup_{k \geq 0} (I : (x_1, \ldots, x_n)^k).$$

A homogeneous ideal I is *m-saturated* if

$$I_d^{sat} = I_d \quad \text{for all } d \geq m.$$

The *saturation degree* of I, denoted $\mathrm{sat}(I)$, is the smallest m for which I is m-saturated.

A homogeneous ideal I is *m-regular* if, in the minimal free resolution of I, for all $p \geq 0$, every p-th syzygy has degree $\leq m + p$. The *regularity* of I, $\mathrm{reg}(I)$, is the smallest such m. The regularity has an alternate description, in terms of cohomology. David Mumford defined the regularity of a coherent sheaf on projective space (now known as Castelnuovo-Mumford regularity) as follows: a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{n-1}$ is said to be *m-regular* if $H^q(\mathbb{P}^{n-1}, \mathcal{F}(m-q)) = 0$ for all $q > 0$; the regularity, $\mathrm{reg}(\mathcal{F})$, is the smallest such m. If I is a saturated ideal, m-regularity in the first sense is equivalent to the geometric condition that the associated sheaf $\mathcal{I}$, on projective space $\mathbb{P}^{n-1}$, satisfies the condition of Casteluovo-Mumford m-regularity. In general case, we may show the following fact with local cohomology [11]:

$$\mathrm{reg}(I) = \max\{\mathrm{sat}(I), \mathrm{reg}(\mathcal{I})\}.$$

A great deal of fundamental information about I can be read off if I is a Borel fixed monomial ideal of R. The following is a useful property of Borel fixed ideals.

Theorem 2.2. ([2], [16]) *For a Borel fixed monomial ideal $I \subset R = K[x_1, \ldots, x_n]$,*

(a) $I^{\mathrm{sat}} = \bigcup_{k=0}^{\infty} (I : x_n^k)$.
(b) $\mathrm{sat}(I) =$ *the maximal degree of generators involving x_n.*
(c) $\mathrm{reg}(I) =$ *the maximal degree of generators of I.*

We introduce a piece of notation. If $K = (k_1, \ldots, k_n)$, we denote by x^K the monomial

$$x^K = x_1^{k_1} \cdots x_n^{k_n}$$

in R, and by $|K|$ its degree $|K| = \sum_{j=1}^{n} k_j$. For monomial x^K with exponent $K = (k_1, \dots, k_n)$ one defines

$$\max(x^K) = \max\{j : k_j > 0\}$$
$$\min(x^K) = \min\{j : k_j > 0\}.$$

For a set of monomials A

$$D(A) = \max\{\min(x^K) : x^K \in A\}$$
$$M(A) = \max\{\max(x^K) : x^K \in A\}$$

If A is the set of the minimal generators of a monomial ideal $I \subset R$, we set

$$D(I) = D(A), \quad M(I) = M(A).$$

Then we get the following lemma.

Lemma 2.3. *If I is a Borel fixed monomial ideal of $R = k[x_1, \dots, x_n]$ such that $n \geq 2$ and $\dim(R/I) > 0$ then,*

(a) $D(I) = \operatorname{codim}(R/I) = n - \dim(R/I)$
(b) $M(I) = \operatorname{codepth}(R/I) = n - \operatorname{depth}(R/I)$.

Proof. By definition of $D(I)$, there is a minimal generator $x^K \in A$ such that $D(I) = \min(x^K)$, where A is the set of the minimal monomial generators of I. Since I is strongly stable, $x_{D(I)}^{|K|}$ must be in I and thus $\sqrt{I} = (x_1, x_2, \cdots, x_{D(I)})$. Since the dimension of R/I is equal to the dimension of $R/\sqrt{I}$, where $\sqrt{I}$ is the radical ideal of I,

$$\dim(R/I) = n - D(I).$$

For the proof of (b), We begin with induction on the number of variables n. In case $n = 2$, it is clear. Suppose that $n > 2$. If $M(I) = n$ then there is a minimal generator of I involving x_n, by definition of $M(I)$. Hence I is not a saturated ideal. Let $H_m^0(R/I)$ be the 0-th local cohomology of R/I and let I^{sat} be the saturation of I. Then

$$H_m^0(R/I) = I^{\mathrm{sat}}/I \neq 0$$

and so $\operatorname{depth}(R/I) = 0$. Suppose that $M(I) < n$. If we consider the homogeneous ideal $J = I + (x_n)/(x_n)$ in $S = k[x_1, \dots, x_{n-1}]$ then $M(I) = M(J)$ and

$$M(J) = (n-1) - \operatorname{depth}(S/J)$$

by the induction hypothesis. Since $M(I) < n$, x_n is a regular element of R/I and $\operatorname{depth}(R/I) = 1 + \operatorname{depth}(S/J) = n - M(I)$. □

From Lemma 2.3, if we have the generic initial ideal of a homogeneous ideal $I \subset R$ under a monomial term order τ then we may know the dimension of R/I. In particular, if we let X be a closed subscheme and let I_X be the defining saturated ideal of X in $\mathbb{P}^{n-1}$, then the codimension of X is the same as $D(\operatorname{Gin}_\tau I_X)$ for any monomial term order τ. Note that $\operatorname{Gin}_\tau(I_X)$ must have a generator of the form $x_{D(I)}^r$ for some positive number r.

Definition 2.4. Let I be a Borel fixed monomial ideal of R and suppose that

$$D(I) < M(I).$$

We denote by $A = \{m_1, \dots, m_s\}$ the set of minimal monomial generators of I. A monomial $x^K \in R$ is said to be a *generalized sporadic zero* of I if it satisfies the

condition that there is a generator $m_i \in A$ satisfying $\max(m_i) = M(I)$, such that $x^r_{M(I)} x^K = m_i$ for some $r > 0$. Now we put:

$$\begin{aligned} \mathrm{Spor}(I) &= \{x^K \in R \,:\, x^K \text{ is a generalized sporadic zero of } I\} \\ \mathrm{Spor}(m, I) &= \{x^K \in R \,:\, x^K \in \mathrm{Spor}(I),\, |K| = m\} \end{aligned}$$

In case $D(I) = M(I)$, we set $\mathrm{Spor}(I) = \emptyset$ for convenience.

Corollary 2.5. *Let I be a Borel fixed monomial ideal. Then the following two conditions are equivalent:*

(a) *R/I is a graded Cohen-Macaulay ring.*
(b) *$D(I) = M(I)$.*

Proof. This follows immediately from Lemma 2.3. □

For a homogeneous ideal I, there is a Borel fixed monomial ideal canonically attached to I: the generic initial ideal $\mathrm{Gin}(I)$ with respect to the reverse lexicographic order. It plays a fundamental role in the investigation of many algebraic, homological, combinatorial and geometric properties of the ideal I itself. From now on, we will only use reverse lexicographic order and we set

$$M(\mathrm{Gin}(I)) = M(I), \quad D(\mathrm{Gin}(I)) = D(I), \quad \mathrm{Spor}(\mathrm{Gin}(I)) = \mathrm{Spor}(I).$$

For many geometric applications, and also for doing inductive arguments, it is useful to know what happens when we restrict to a generic hyperplane. Let h be a general linear form. Then we may consider $J = (I + (h))/(h)$ as a homogeneous ideal of $S = k[x_1, \dots, x_{n-1}]$. One has the following well known fact [2, 16]:

$$\mathrm{Gin}(J) = \mathrm{Gin}(I)|_{x_n \to 0}. \tag{2.1}$$

Since $\mathrm{Gin}(I^{sat}) = \mathrm{Gin}(I)|_{x_n \to 1}$ for any homogeneous ideal I, we get

$$\mathrm{Gin}(J^{sat}) = (\mathrm{Gin}(I)|_{\,x_n \to 0})|_{x_{n-1} \to 1}. \tag{2.2}$$

Lemma 2.6. *Let X be a closed subscheme in $\mathbb{P}^{n-1}$ and let I_X be the defining saturated ideal of X. For a general linear form h and for any positive integer $m \geq 1$, we denote by $K(m, I_X)$ the kernel of the map*

$$H^1(\mathcal{I}_X(m-1)) \xrightarrow{\cdot h} H^1(\mathcal{I}_X(m))$$

Then, the following holds:

$$\dim_k K(m, I_X) = \begin{cases} |\mathrm{Spor}(m, I_X)| & \text{if } M(I_X) = n-1; \\ 0 & \text{if } M(I_X) < n-1. \end{cases}$$

Proof. Let $m \geq 1$ and we set

$$J = \left(\frac{I_X + (h)}{(h)} \right).$$

Then, from (2.1), (2.2) and from the long exact sequence

$$0 \to I_X(m-1) \xrightarrow{\cdot h} I_X(m) \to I_{X \cap H}(m) \to H^1(\mathcal{I}_X(m-1)) \xrightarrow{\cdot h} H^1(\mathcal{I}_X(m)) \to \cdots$$

we get the desired result:

$$\begin{aligned}\dim_k(K(m, I_X)) &= \dim_k(J^{sat}/J)_m \\ &= \dim_k \frac{((\mathrm{Gin}(I_X)|_{x_n\to 0})|_{x_{n-1}\to 1})_m}{(\mathrm{Gin}(I_X)|_{x_n\to 0})_m} \\ &= \begin{cases} |\mathrm{Spor}(m, I_X)| & \text{if } M(I_X) = n-1; \\ \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

□

The following theorem gives the relation between the existence of a generalized sporadic zero and the nonvanishing of sheaf cohomology of an ideal sheaf $\mathcal{I}_X$.

Theorem 2.7. *Let X be the closed subscheme in $\mathbb{P}^{n-1}$ and let I_X be the defining saturated ideal of X. Set $d = n - M(I_X)$ and assume that* $\mathrm{Spor}(m, I_X) \neq \emptyset$ *for some $m \geq 1$. Then,*

(a) $H^i(\mathcal{I}_X(j)) = 0$ *for* $0 < i < d$ *and* $j \in \mathbb{Z}$.
(b) $H^d(\mathcal{I}_X(m-d)) \neq 0$.

Proof. To prove this assertion, we use induction on the dimension of the ambient projective space. Note that $d > 0$ if I_X is a saturated ideal. In case $d = 1$, the result follows from Lemma 2.6.

For $d > 1$, we will first prove (a). Note that $M(I_X) < n-1$. By Lemma 2.6, the map

$$H^1(\mathcal{I}_X(j-1)) \to H^1(\mathcal{I}_X(j))$$

is an inclusion for all integers j, and hence $H^1_*(\mathcal{I}_X) = 0$. This gives the proof for $d = 2$. Now assume that $d > 2$. From the fact that $M(I_X) = M(I_{X\cap H})$ for a general hyperplane H (it follows from Lemma 2.3), we have $d - 1 = (n-1) - M(I_{X\cap H})$. Hence

$$H^i(\mathcal{I}_{X\cap H}(j)) = 0 \quad \text{for } 0 < i < d-1$$

by the induction hypothesis. Then it follows from the exact sequence

$$\cdots \to H^{i-1}(\mathcal{I}_{X\cap H}(j)) \to H^i(\mathcal{I}_X(j-1)) \to H^i(\mathcal{I}_X(j)) \to$$

that $H^i(\mathcal{I}_X(j)) = 0$ for $0 < i < d$ and for all j. This proves (a).

We now prove (b). For $d > 1$, note again that $M(I_X) = M(I_{X\cap H})$. Hence we have

$$H^{d-1}(\mathcal{I}_{X\cap H}(m-d+1)) \neq 0$$

by the induction hypothesis. From the exact sequence

$$0 \to H^{d-1}(\mathcal{I}_{X\cap H}(m-d+1)) \to H^d(\mathcal{I}_X(m-d)) \to H^d(\mathcal{I}_X(m-d+1))$$

we know that $H^d\big(\mathcal{I}_X(m-d)\big)$ does not vanish, and this proves (b). □

From this, we give a cohomological proof of the following well known fact.

Corollary 2.8. ([10]) *For I be a homogeneous ideal of $R = K[x_1, \ldots, x_n]$, consider the generic initial ideal for the degree reverse lexicographic order. Then,*

$$\mathrm{depth}(R/I) = \mathrm{depth}\left(R/\mathrm{Gin}(I)\right).$$

Proof. Let $H^i_m(R/I)$ be the local cohomology of R/I. By Lemma 2.3, we know that

$$\operatorname{depth}(R/\operatorname{Gin}(I)) = n - M(I),$$

so it is enough to show that

$$n - M(I) = \min\{i : H^i_m(R/I) \neq 0\}.$$

But this follows from Theorem 2.7 and the following fact ([11]):

$$H^i_m(R/I) = \bigoplus_{j\in\mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{I}(j))$$

for $0 < i < n$. $\square$

Corollary 2.9. *Let I be a homogeneous ideal of $R = K[x_1, \ldots, x_n]$. Then the following facts are equivalent.*

(a) *R/I is Cohen-Macaulay.*
(b) *$R/\operatorname{Gin}(I)$ is Cohen Macaulay.*
(c) *$M(I) = D(I)$.*

Example 2.10. We consider the homogeneous ideal in $k[x_1, x_2, x_3, x_4]$

$$I = (x_3^3 - x_1x_4^2, x_1^2x_3^2 - x_2^3x_4, x_2^3x_3 - x_1^3x_4, x_2^6 - x_1^5x_3).$$

Using Macaulay2, we get the generic initial ideal of I

$$\operatorname{Gin}(I) = (x_1^3, x_1^2x_2^2, x_1x_2^3, x_2^5, x_2^4x_3^2)$$

under reverse lexicographic order. Then we know that I is a saturated ideal, $D(I) = 2$ and $M(I) = 3$. Hence I is the defining ideal of a projective curve C in $\mathbb{P}^3$. Note that $\operatorname{Spor}(I) = \{x_2^4, x_2^4x_3\}$ and C is not arithmetically Cohen-Macaulay.

Remark 2.11. If we use another monomial term order then $M(I)$ may change, but $D(I)$ is independent of the monomial order.

By Theorem 2.2, the regularity of $\operatorname{Gin}(I)$ is the largest degree of a generator of $\operatorname{Gin}(I)$. Bayer and Stillman [2] showed the regularity of I is equal to the regularity of $\operatorname{Gin}(I)$.

Theorem 2.12. ([2], [16], [17]) *For any homogeneous ideal I, using the reverse lexicographic order,*

$$\operatorname{sat}(I) = \operatorname{sat}(\operatorname{Gin}(I)),$$
$$\operatorname{reg}(I) = \operatorname{reg}(\operatorname{Gin}(I)).$$

For a homogeneous ideal $I \subset R$ there exists a flat family of ideals I_t with $I_0 = \operatorname{in}(I)$ and I_t canonically isomorphic to I for all $t \neq 0$ (Corollary 1.21 in [16]). Using this result we know the minimal free resolution of I is obtained from that of $\operatorname{in}(I)$ by cancelling some adjacent terms of the same degree. That is, we can always choose a complex of $K \cong R/m-$modules $V^d_\bullet$ such that

$$V^d_i \cong \operatorname{Tor}^R_i(\operatorname{in}(I), K)_d$$
$$H_i(V^d_\bullet) \cong \operatorname{Tor}^R_i(I, K)_d.$$

Although the minimal free resolution of a general monomial ideal can be quite complicated, the situation for Borel fixed monomial ideals is very nice. Eliahou and Kervaire [13] gave a result for the structure of the minimal free resolution of a Borel fixed ideal. Using these results, we have the following.

Theorem 2.13. (Crystallization Principle) *Let I be a homogeneous ideal generated in degrees $\leq d$. Assume that there is a monomial order τ such that $\mathrm{Gin}_\tau(I)$ has no generator in degree $d+1$. Then $\mathrm{Gin}_\tau(I)$ is generated in degrees $\leq d$ and I is d-regular.*

Proof. The case of arbitrary monomial order τ can be proved in the same manner as the proof of Theorem 2.28 in [16]. □

Remark 2.14. We can consider Theorem 2.13 as a generalization of the Gotzmann persistence theorem (Theorem 3.8 in [16]). Let I^{lex} be the lex-segment ideal of I. We know that the lex-segment ideal has the largest Betti numbers in the class of the ideals with a given Hilbert function (Theorem 2 in [3]). Then $\mathrm{Gin}_\tau(I)$ has no generator in degree $d+1$ for every monomial order τ if I^{lex} has no generator in degree $d+1$, which is equivalent to the Hilbert function of R/I has maximal growth in degree d. Hence if a homogeneous ideal I is generated in degrees $\leq d$ and the Hilbert function of R/I has maximal growth in degree d then I is d-regular. We will use Theorem 2.13 to generalize results in [4].

For a homogeneous ideal $I \subset R = K[x_1, \ldots, x_n]$, let $d = \dim(R/I)$. L.T. Hoa and N. V. Trung defined the *s-reduction number* of R/I for $s \geq d$ in [19]. They have shown that $r_s(R/I) = r_s(R/\mathrm{Gin}(I))$ (Theorem 2.3 in [19]). If I is a Borel fixed monomial ideal we know that there are positive numbers $t_1, \ldots, t_d$ such that $x_i^{t_i}$ is a minimal generator of $\mathrm{Gin}(I)$. In [19], authors also proved that if a monomial ideal I is strongly stable, then

$$r_s(R/I) = \min\{k \,:\, x_{n-s}^{k+1} \in I\}. \tag{2.3}$$

From these facts we get the following Lemma. We will take this lemma as the definition of $r_s(R/I)$ for our purposes.

Lemma 2.15. *For a homogeneous ideal I of R and for $s \geq \dim(R/I)$, the s-reduction number $r_s(R/I)$ can be given as the following:*

$$\begin{aligned} r_s(R/I) &= \min\{k \,:\, x_{n-s}^{k+1} \in \mathrm{Gin}(I)\} \\ &= \min\{k \,:\, \text{Hilbert function of } R/(I+J) \text{ vanishes in degree } k+1 \,\} \end{aligned}$$

where J is generated by s general linear forms of R.

Proof. The first part of Lemma follows directly from results in [19]. Consider a homogeneous ideal J generated by s general linear forms. Then, for the reverse lexicographic order, we know that

$$M := \mathrm{Gin}\left(\frac{I+J}{J}\right) = \frac{\mathrm{Gin}(I) + (x_{n-s+1}, \ldots, x_n)}{(x_{n-s+1}, \ldots, x_n)}$$

from (2.1). Hence, for $s \geq d$, we can compute $r_s(R/I)$ from (2.3) by looking for the smallest degree in which the monomial ideal M becomes the unique maximal ideal of $R/(x_{n-s+1}, \ldots, x_n)$ by the strongly stable property of M. This proves the second part of the Lemma since the Hilbert function of $(I+J)/J$ has the same Hilbert function as M. □

Remark 2.16. Let Γ be a finite set of points (or more generally, any zero-dimensional scheme). There is a strong connection between the definition of $r_2(R/I_\Gamma)$ and the Weak Lefschetz Property (WLP) (cf. [18]). We first recall this property. Let $J = (L_1, L_2)$ be generated by general linear forms. Let $K = I_\Gamma + (L_1)$,

and let $A = R/K$ be the Artinian reduction of R/I_Γ by L_1. Then multiplication by L_2 gives an exact sequence

$$0 \to \left(\frac{[K : L_2]}{K}\right)_d \to (R/K)_d \xrightarrow{\times L_2} (R/K)_{d+1} \to R/(I_\Gamma + J)_{d+1} \to 0.$$

The Weak Lefschetz Property merely says that this multiplication has maximal rank, for all d. Now, computing $r_2(R/I_\Gamma)$ amounts to studying the surjectivity of $\times L_2$ above, which is a triviality when we know maximal rank. Indeed, WLP says that there is an expected value for $r_2(R/I_\Gamma)$, and that this value is achieved.

Intuitively, WLP should be true "most of the time." Attempts to make this more precise were given in [18] and in [21]. A very natural question, then, is whether WLP should hold for points with the Uniform Position Property (UPP). In Example 6.9 we will show that this is not the case.

In the case of low dimensional schemes, the following Lemma 2.17 and Proposition 2.18 show a relation between the reduction number and the regularity.

Lemma 2.17. *Let $I \subset R = K[x_1, \ldots, x_n]$ be a saturated ideal and suppose that $\dim(R/I) = 1$. Then the regularity of I is equal to $r = r_1(R/I) + 1$.*

Proof. Consider the generic initial ideal $\mathrm{Gin}(I)$ in terms of reverse lexicographic order. Then x_{n-1}^r is a minimal generator of $\mathrm{Gin}(I)$ (Lemma 2.15). We know that all monomials of degree r that do not involve the variable x_n are contained in $\mathrm{Gin}(I)$ by the strongly stable property. Hence if there is a minimal generator of $\mathrm{Gin}(I)$ in degree $> r$ then it should involve the variable x_n. But this is impossible because $\mathrm{Gin}(I)$ is a saturated Borel fixed monomial ideal (Theorem 2.2 and Theorem 2.12). This implies that the maximal degree of a minimal generator of $\mathrm{Gin}(I)$ is r, which is equal to the regularity of I by Theorem 2.12. $\square$

Proposition 2.18. *Let C be a projective curve in $\mathbb{P}^{n-1}$, $n > 3$ (not necessarily reduced and irreducible). Suppose that there is an integer $d > r_2(R/I_C)$ such that*

$$h^1(\mathcal{I}_C(d-1)) = 0.$$

Then I_C is d-regular.

Proposition 2.18 shows when we can get information about the regularity of a projective curve $C \subset \mathbb{P}^{n-1}$ by the vanishing of $h^1(\mathcal{I}_C(t))$. Normally it requires some knowledge of $h^2(\mathcal{I}_C(t))$ as well.

Proof. We will prove that $h^2(\mathcal{I}_C(d-2)) = 0$. For a general hyperplane H,

$$r_1(I_{C\cap H}) \le r_2(I_C)$$

since $\mathrm{Gin}(I_{C\cap H}) = (\mathrm{Gin}(I_C)|_{x_n \to 0})|_{x_{n-1}\to 1}$ (see Proposition 2.15). Therefore we have that

$$\mathrm{reg}(I_{C\cap H}) \le r_2(R/I_C) + 1$$

by Lemma 2.17, and so

$$h^1(\mathcal{I}_{C\cap H}(t)) = 0$$

for all $t \ge r_2(R/I_C)$. Consider the following long exact sequence

$$\longrightarrow (h^1(\mathcal{I}_{C\cap H}(t)) \longrightarrow h^2(\mathcal{I}_C(t-1)) \longrightarrow h^2(\mathcal{I}_C(t)) \longrightarrow,$$

then we get

$$0 \longrightarrow h^2(\mathcal{I}_C(t-1)) \longrightarrow h^2(\mathcal{I}_C(t))$$

is injective for all $t \geq r_2(R/I_C)$ and so $h^2(\mathcal{I}_C(t-1)) = 0$ for all $t \geq r_2(R/I_C)$. This implies that $h^2(\mathcal{I}_C(d-2)) = 0$ and I_C is d-regular. $\square$

3. The regularity of a Cohen-Macaulay ring

Let X be a closed subscheme of $\mathbb{P}^{n-1}$ which is not contained in any hyperplane. Suppose that I_X is the defining saturated ideal of X and that R/I_X is Cohen-Macaulay. Then the following is a well-known fact ([11] Corollary 4.14, [12]):

$$\operatorname{reg}(I_X) \leq \deg(X) - \operatorname{codim}(X) + 1.$$

In this section, we will give a stronger bound, by using the initial degree of I_X (Theorem 3.3). We start with the following:

Proposition 3.1. *Let I be a saturated Borel fixed monomial ideal of $R = K[x_1, \ldots, x_n]$ with $\dim(R/I) = 1$. Then,*

$$\operatorname{reg} I \leq \deg(R/I) - \binom{\alpha - 1 + n - 1}{\alpha - 1} + \alpha$$

where $\alpha = \min\{l \mid I_l \neq 0\}$ is the initial degree of I.

Proof. Since R/I is a Cohen-Macaulay ring,

$$D(I) = M(I) = n - 1.$$

Hence, there exist positive numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$ such that

$$I = (x_1^{\alpha}, \ldots, x_2^{\lambda_1}, \ldots, x_{n-2}^{\lambda_{n-2}}, \ldots, x_{n-1}^{\lambda_{n-1}}).$$

Clearly if $J = (x_1, \ldots, x_{n-1})^{\alpha}$, then

$$\deg R/J = \binom{\alpha + n - 1}{\alpha} - \binom{\alpha + n - 2}{\alpha} = \binom{\alpha - 1 + n - 1}{\alpha - 1}.$$

Now, let J' be the ideal generated by $x_{n-1}^{\lambda_{n-1}}$ and by all the monomials of degree α except x_{n-1}^{α} in the variables $x_1 \ldots, x_{n-1}$:

$$J' = (x_1^{\alpha},\ x_1^{\alpha-1}x_2,\ \ldots, x_{n-2}x_{n-1}^{\alpha-1},\ x_{n-1}^{\lambda_{n-1}}).$$

We first claim that

$$\deg(R/J') = \binom{\alpha - 1 + n - 1}{\alpha - 1} - \alpha + \lambda_{n-1}. \tag{3.1}$$

Indeed, if we consider the first difference Hilbert function (i.e h-vector) of R/J', then $\Delta H(R/J', t) = 1$ if $\alpha \leq t < \lambda_{n-1}$, since J' is defined by using all the monomials of degree α except x_{n-1}^{α} and we have reduced modulo a general linear form (which we can take to be x_n). Since

$$H(R/J', \alpha) = H(R/J, \alpha) + 1 = \binom{\alpha - 1 + n - 1}{\alpha - 1} + 1,$$

(3.1) follows from

$$\deg(R/J') = H(R/J', \lambda_{n-1} - 1) = H(R/J', \alpha) + \sum_{t=\alpha+1}^{\lambda_{n-1}-1} (\Delta H(R/J', t)).$$

Note also that

$$\deg(R/J') \leq \deg(R/I). \tag{3.2}$$

Since I and J' are also Borel fixed monomial ideals, by virtue of Theorem 2.2,

$$\lambda_{n-1} = \operatorname{reg}(I) = \operatorname{reg}(J')$$

and it follows from (3.1) and (3.2) that

$$\operatorname{reg}(I) \leq \deg(R/I) - \binom{\alpha - 1 + n - 1}{\alpha - 1} + \alpha$$

as claimed. □

Corollary 3.2. *If I is a Borel fixed monomial ideal such that R/I is Cohen-Macaulay with* $\dim R/I \geq 1$ *then*

$$\operatorname{reg}(I) \leq \deg(R/I) - \binom{\alpha - 1 + D(I)}{\alpha - 1} + \alpha$$

where $\alpha = \alpha(I) = \min\{l \mid I_l \neq 0\}$.

Proof. To get the regularity bound, we work by induction on n. If $D(I) = n - 1$ then it is proved by Proposition 3.1. Now suppose that $D(I) < n-1$. If we consider the ideal $J = I + (x_n)/(x_n)$ of $S = k[x_1, \dots, x_{n-1}]$, then J is also a Borel fixed monomial ideal such that

$$\operatorname{reg}(I) = \operatorname{reg}(J), \quad D(I) = D(J)$$

and

$$\deg(R/I) = \deg(S/J), \quad \alpha(I) = \alpha(J).$$

Since S/J is also a Cohen-Macaulay ring, by the induction hypothesis,

$$\begin{aligned}\operatorname{reg}(I) = \operatorname{reg}(J) &\leq \deg(R/J) - \binom{\alpha(J) - 1 + D(J)}{\alpha(J) - 1} + \alpha(J) \\ &= \deg(R/I) - \binom{\alpha(I) - 1 + D(I)}{\alpha(I) - 1} + \alpha(I)\end{aligned}$$

as desired. □

The following is the main result of this section. It is a special case of Satz 3 of [22], taking $I(A) = 0$ since X is arithmetically Cohen-Macaulay. However, our method of proof is quite different.

Theorem 3.3. *Let X be an arithmetically Cohen-Macaulay subscheme of $\mathbb{P}^{n-1}$ with* $\operatorname{codim}(X) = e$*. Then,*

$$\operatorname{reg}(I_X) \leq \deg(X) - \binom{\alpha - 1 + e}{\alpha - 1} + \alpha$$

where $\alpha = \alpha(I_X)$ is the initial degree of I_X.

Proof. It follows from Lemma 2.3, Corollary 2.9 and Corollary 3.2, since $\alpha(I_X) = \alpha(\operatorname{Gin}(I_X))$. □

Remark 3.4. Under the assumption of Theorem 3.3, consider the function $f(x) = \binom{x-1+e}{x-1} - x$. If there is no linear form in I_X, we have:

$$\begin{aligned}\operatorname{reg}(I_X) &\leq \deg(R/I_X) - f(\alpha(I_X)) \\ &\leq \deg(R/I_X) - f(2) \\ &= \deg(R/I_X) - D(I_X) + 1 \\ &= \deg(R/I_X) - \operatorname{codim}(R/I_X) + 1.\end{aligned}$$

since $f(x)$ is strictly increasing for $x \geq 2$. Hence we may think of Theorem 3.3 as a generalization of a result in [11].

4. Application to the Hilbert functions of a set of points

A. Bigatti, A.V. Geramita and J.C. Migliore gave many geometric results relating to the maximal growth of the first difference of Hilbert function in [4]. In particular, they considered a reduced set of points Γ in $\mathbb{P}^{n-1}$ and proved several geometric consequences of the condition

$$\Delta H(\Gamma, d) = \Delta H(\Gamma, d+1) = s \tag{4.1}$$

for $d \geq s$. In this section, we generalize these results of [4] by assuming only $d > r_2(R/I_Z)$. In the next section we will take up the question of uniform position, and see what can be said (and what can *not* be said) with that additional hypothesis. We will see that before we add the uniformity assumption, virtually everything that holds in the case $d \geq s$ ([4] Theorem 3.6) remains true here (Theorem 4.6). The only difference will be our inability here to prove a reducedness result (see Remark 4.7). We start with the following.

Proposition 4.1. *Let Y be a scheme of dimension ≤ 1 in $\mathbb{P}^{n-1}$, $n \geq 3$. Then*

$$\Delta H(R/I_Y, d) \geq \Delta H(R/I_Y, d+1) \tag{4.2}$$

for $d \geq r_2(R/I_Y)$. Moreover, for $d > r_2(R/I_Y)$, we have equality in (4.2) *if and only if* $\mathrm{Gin}(I_Y)$ *has no minimal generators in degree $d+1$.*

Proof. Let $J = (L_1, L_2)$ be generated by general linear forms. Let $K = I_Y + (L_1)$; this can be viewed as a homogeneous ideal in $S = R/(L_1)$. Let $A = S/K$. Then multiplication by L_2 gives an exact sequence

$$0 \to \left(\frac{[K : L_2]}{K}\right)_d \to (S/K)_d \xrightarrow{\times L_2} (S/K)_{d+1} \to (R/(I_\Gamma + J))_{d+1} \to 0. \tag{4.3}$$

By the definition of $r_2(R/I_Y)$,

$$(R/(I_\Gamma + J))_{d+1} = 0$$

for $d \geq r_2(R/I_Y)$. Since I_Y is a saturated ideal, $\Delta H(R/I_Y, d)$ is the same as the Hilbert function $H(S/K, d)$ for all d. This proves the first part of the theorem.

Now consider the following exact sequence:

$$0 \to \left(\frac{[K : L_2]}{K}\right)_d \to (S/K)_d \xrightarrow{\times L_2} (S/K)_{d+1} \to 0 \tag{4.4}$$

for $d > r_2(R/I_Y)$. Note that $r_s(R/I_Y) = r_s(R/\mathrm{Gin}(I_Y))$ by Theorem 2.3 in [19]. So we may replace K, L_1 and L_2 by $\mathrm{Gin}(K)$, x_n and x_{n-1} respectively and thus reduce to the case where K is a Borel fixed monomial ideal, since we know that (2.1) and (2.2) hold under reverse lexicographic order (note that K and $\mathrm{Gin}(K)$ have the same Hilbert function).

Since I_Y is a saturated ideal, $\mathrm{Gin}(K)$ have no minimal generator of degree $d+1$ in $S = R/L_1$ if and only if $\mathrm{Gin}(I_Y)$ have no minimal generator of degree $d+1$ in R for $d > r_2(R/I_Y)$. For the proof of the second part, therefore, it is enough to show that $\mathrm{Gin}(K)$ has no minimal generators in degree $d+1$ if and only if

$$\left(\frac{[\mathrm{Gin}(K) : x_{n-1}]}{\mathrm{Gin}(K)}\right)_d = 0 \tag{4.5}$$

from the exact sequence (4.4).

Now assume that (4.5) holds. We know that $x_{n-2}^d \in \mathrm{Gin}(K)$ by the first equality of Lemma 2.15. This implies that $\mathrm{Gin}(K)_d$ contains each monomial x^J of degree d such that

$$\mathrm{supp}(J) \subset \{x_1, \ldots, x_{n-2}\}$$

since $\mathrm{Gin}(K)$ is a Borel fixed monomial ideal. If $\mathrm{Gin}(K)$ has a minimal generator of degree $d+1$, then it must involve the variable x_{n-1}. But this is impossible because it means $([\mathrm{Gin}(K) : x_{n-1}]/\mathrm{Gin}(K))_d \neq 0$. Hence $\mathrm{Gin}(K)$ has no minimal generators in degree $d+1$.

Conversely, if

$$([\mathrm{Gin}(K) : x_{n-1}]/\mathrm{Gin}(K))_d \neq 0$$

then there exists a monomial x^L such that $x_{n-1}x^L \in \mathrm{Gin}(K)_{d+1}$ but $x^L \notin \mathrm{Gin}(K)_d$. Assume that $x_{n-1}x^L$ is not a minimal generator of $\mathrm{Gin}(K)$. Then we can choose a monomial $x^J \in \mathrm{Gin}(K)_d$ of degree d, satisfying

$$x_i x^J = x_{n-1}x^L$$

for $1 \leq i < n-1$. Note that x^J should contain the variable x_{n-1}. Hence we have

$$x^L = x^J\left(\frac{x_i}{x_{n-1}}\right) \in \mathrm{Gin}(K)_d$$

by the Borel fixed property (or strongly stable) and this contradicts the choice of x^L. Hence the monomial $x_{n-1}x^L$ is a minimal generator of degree $d+1$ of $\mathrm{Gin}(K)$. □

Theorem 4.2. *Let Y be a scheme of dimension ≤ 1 in $\mathbb{P}^{n-1}$, $n \geq 3$. Assume that for some $d > r_2(R/I_Y)$*

$$\Delta H(R/I_Y, d) = \Delta H(R/I_Y, d+1)) = s \neq 0.$$

Then $\langle (I_Y)_{\leq d}\rangle$ is d-regular and it is a saturated ideal defining a one dimensional scheme of degree s in $\mathbb{P}^{n-1}$.

Proof. Let $\bar{I} = \langle (I_Y)_{\leq d}\rangle$. To show that $\bar{I}$ is a saturated ideal, we need only to prove that $\mathrm{Gin}(\bar{I})$ does not have a minimal generator containing the variable x_n (Theorem 2.2). By Theorem 2.13 and Theorem 4.1, $\bar{I}$ is d-regular, and thus $\mathrm{Gin}(\bar{I})$ has no minimal generator in degrees $> d$. Since I_Y is a saturated ideal, $\mathrm{Gin}(I_Y)$ does not have a minimal generator involving x_n. Then it follows from

$$(\mathrm{Gin}(I_Y))_k = (\mathrm{Gin}(\bar{I}))_k$$

for $k \leq d$ that there is no minimal generator of $\mathrm{Gin}(\bar{I})$ containing the variable x_n in degree $\leq d$. Hence $\bar{I}$ is a saturated ideal.

Since $(\bar{I})_i = (I_Y)_i$ for $i \leq d+1$, we know that

$$r_2(R/\bar{I}) = r_2(R/I_Y)$$

and that $\bar{I}$ defines a scheme of dimension ≤ 1. Note that $\mathrm{Gin}(\bar{I})$ does not have minimal generator of degree $k > d$. Applying Theorem 4.1 repeatedly, we get

$$\Delta H(R/\bar{I}, i) = \Delta H(R/\bar{I}, i+1)) = s \neq 0$$

for all $i \geq d > r_2(R/\bar{I})$. This means that $\bar{I}$ defines a one dimensional subscheme in $\mathbb{P}^{n-1}$ of degree s. □

Corollary 4.3. *Let Γ be a set of points in $\mathbb{P}^{n-1}$, $n \geq 3$, and let $(1, h_1, \ldots, h_t)$ be the h-vector of R/I_Γ. Then $h_d \geq h_{d+1}$ for $d > r_2(R/I_\Gamma)$. Suppose that*

$$h_d = h_{d+1} = s$$

for $d > r_2(R/I_\Gamma)$. Then $(I_\Gamma)_{\leq d}$ is a saturated ideal defining a one dimensional subscheme of degree s in $\mathbb{P}^{n-1}$ and it is d-regular.

Proof. Since Γ is arithmetically Cohen-Macaulay,

$$\Delta H(R/I_\Gamma, d) = h_d.$$

Hence the result follows from Corollary 4.2. □

Remark 4.4. Let $Y \subset \mathbb{P}^{n-1}$ be a subscheme of any dimension, $n \geq 3$. Assume that

$$\Delta H(Y, d) = \Delta H(Y, d+1) = s \neq 0$$

for some $d \geq s$. Then $d > r_2(R/I_Y)$ and $\dim(Y) \leq 1$. Indeed, the condition $d \geq s$ means that the d-binomial expansion of s is $\binom{d}{d} + \cdots + \binom{d-s+1}{d-s+1}$ and so we have maximal growth of the Hilbert function of $R/(I_Y + (L))$ in degree d, where L is a general linear form of R. By Theorem 3.6 of [4], this implies that $(I_Y)_{\leq d}$ is a saturated ideal defining a one dimensional subscheme in $\mathbb{P}^{n-1}$. Then

$$D((I_Y)_{\leq d}) = n - 2$$

by Lemma 2.3, and x_{n-2}^d is contained in $(I_Y)_{\leq d}$. This implies that $d > r_2(R/I_Y)$ and that $D(I_Y) \geq n-2$, so that $\dim(Y) \leq 1$. Hence we can think of the condition $d > r_2(R/I_Y)$ as generalizing the assumption $d \geq s$ which is in [4].

Remark 4.5. Since Corollary 4.3 weakens the hypothesis $d \geq s$ of [4], one might ask if our hypothesis $d > r_2(R/I)$ can be weakened even further. We now show by simple examples that this is not the case for results involving the first difference, although in section 6 we do weaken this hypothesis by invoking the second difference.

Let Γ be a complete intersection in $\mathbb{P}^3$ of type $(4, 4, 4)$. The first difference of the Hilbert function of Γ is

$$1 \quad 3 \quad 6 \quad 10 \quad 12 \quad 12 \quad 10 \quad 6 \quad 3 \quad 1.$$

Since the Artinian reduction of R/I_Γ has the Weak Lefschetz property ([18] Theorem 2.3), we see that $r_2(R/I_\Gamma) = 4$. Since $\Delta H(R/I_\Gamma, 4) = \Delta H(R/I_\Gamma, 5) = 12$, but clearly the component $(I_\Gamma)_4$ defines a zeroscheme (namely Γ) rather than a curve, we see that we cannot weaken the assumption $d > r_2(R/I_\Gamma)$. Of course such an example can be constructed in any codimension ≥ 3.

Similarly, we have the following series of examples (and again, the same sort of thing could be done in higher projective space):

7 general points in $\mathbb{P}^3$ have h-vector 1 3 3
16 general points in $\mathbb{P}^3$ have h-vector 1 3 6 6
30 general points in $\mathbb{P}^3$ have h-vector 1 3 6 10 10
etc.

In each case, the value of $r_2(R/I_\Gamma)$ is the "expected" one, and we have $d = r_2(R/I_\Gamma)$, but the component of I_Γ in degree d is zero, while the component of I_Γ in degree $d+1$ defines a zero-dimensional scheme rather than a curve. See also Example 6.9.

We also remark that in general $r_2(R/I_Z)$ can be much smaller than s. For example, let Z be a set of sufficiently many points on a smooth arithmetically Cohen-Macaulay curve of degree 28 in $\mathbb{P}^3$ with h-vector $(1,2,3,4,5,6,7)$. Then it is not hard to show that $r_2(R/I_Z)=6$ while $s=28$.

The next result is our analog to [4] Corollary 3.7.

Theorem 4.6. *Let $Y \subset \mathbb{P}^{n-1}$, $n \geq 3$ be a reduced scheme of any dimension. Assume that $\Delta H(Y,d)=s$ and that the saturated ideal $\langle (I_Y)_{\leq d}\rangle^{\mathrm{sat}}$ defines a curve V of degree s. Then*

(a) $\dim(Y) \leq 1$.
(b) $\langle (I_Y)_{\leq d}\rangle$ *is the saturated ideal of a curve V of degree s (not necessarily unmixed).*

Let C be the unmixed one-dimensional component of the curve V in (b). Let Y_1 be the subvariety of Y on C and Y_2 the "residual" subvariety.

(c) $\langle (I_{Y_1})_{\leq d}\rangle = I_C$.
(d) $\dim(Y_2)=0$ *and* $H(Y_1,t)=H(Y,t)-|Y_2|$ *for all* $t \geq d-1$.
(e)
$$\Delta H(Y_1,t)=\begin{cases}\Delta H(C,t), & \text{for } t \leq d+1\\ \Delta H(Y,t), & \text{for } t \geq d.\end{cases}$$
In particular, $\Delta H(Y_1,t) \geq \Delta H(Y_1,t+1)$ *for all* $t \geq d$.
(f) *If we assume* $h^1(\mathcal{I}_{C_{\mathrm{red}}}(d-1))=0$ *then V is reduced and $C=C_{red}$ is d-regular.*

Proof. The proof is almost same as that of Theorem 3.6 in [4]. We sketch the proof (see [4] for more details). Let $\bar{I}=\langle (I_Y)_{\leq d}\rangle$. If L is a general linear form and H is the corresponding hyperplane, let

$$J=\bar{I}+(L)/(L).$$

Then J^{sat} is the defining ideal of the set of points $I_{V\cap H}$ of degree s in $S=R/(L)$. By the assumption that $\Delta H(Y,d)=s$, we get

$$\Delta H(R/\bar{I},d)=H(S/J,d)=\deg(S/J)=s.$$

This implies $d \geq \operatorname{reg}(J)$. Hence J is saturated in degree $\geq d$ and $H(S/J,k)=s$ for all $k \geq d$. Then, by Lemma 2.17

$$d \geq \operatorname{reg}(J) \geq \operatorname{reg}(J^{\mathrm{sat}})=r_1(S/J^{\mathrm{sat}})+1$$

and thus x_{n-2}^d is contained in $\mathrm{Gin}(J^{\mathrm{sat}})_d=\mathrm{Gin}(J)_d$. Hence $d \geq r_1(S/J)+1= r_2(R/\bar{I})+1$ so $\bar{I}=\langle (I_Y)_{\leq d}\rangle$ is a saturated ideal defining a one dimensional scheme V of degree s in $\mathbb{P}^{n-1}$ by Corollary 4.2. Then $\dim(Y)\leq 1$ follows from $D(I_Y)\geq n-2$. This proves (a) and (b). Let C be the unmixed one-dimensional component of V with degree s. Then the scheme Y is the union of Y_1, which is the subvariety of Y lying on C, and Y_2, the remaining subvariety of Y. Since $I_V=\langle (I_Y)_{\leq d}\rangle$ is d-regular (Theorem 4.2), the same proof as in Theorem 3.3 and Theorem 3.6 of [4] gives

$$h^1(\mathcal{I}_{Y_2}(t))=0 \text{ for all } t \geq d-1 \tag{4.6}$$

$$h^1(\mathcal{I}_{C\cup Y_2}(t))=0 \text{ for all } t \geq d-1 \tag{4.7}$$

$$h^1(\mathcal{I}_{C}(t))=0 \text{ for all } t \geq d-1. \tag{4.8}$$

Since Y_2 is a zero dimensional scheme, we have that $\Delta H(Y_2, t) = 0$ for all $t \geq d$. We also have

$$H(C \cup Y_2, t) = H(Y, t) + H(C, t) - H(Y_1, t) \tag{4.9}$$

from the short exact sequence

$$0 \longrightarrow (I_{C\cup Y_2})_t \longrightarrow (I_Y)_t \oplus (I_C)_t \longrightarrow (I_{Y_1})_t \longrightarrow 0.$$

On the other hand, from the exact sequence

$$0 \longrightarrow \mathcal{I}_{C\cup Y_2} \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow \mathcal{O}_{C\cup Y_2} \longrightarrow 0$$

(respectively the same sequence with $C \cup Y_2$ replaced by C), we get from (4.7) and (4.8) that $H(C \cup Y_2, t) = h^0(\mathcal{O}_C(t)) + |Y_2|$ and $h^0(\mathcal{O}_C(t)) = H(C, t)$ for $t \geq d-1$. Now combining this information and substituting it in (4.9) we get

$$H(Y_1, t) = H(Y, t) - |Y_2|$$

for all $t \geq d-1$. This proves (d). In particular,

$$\Delta H(Y_1, t) = \Delta H(Y, t) \text{ for all } t \geq d.$$

And this proves the second half of the Hilbert function claimed in (e). Notice that we still have the following condition for Y_1:

$$\Delta H(Y_1, d) = \Delta H(Y_1, d+1)$$

Since $I_Y \subset I_{Y_1}$, clearly $r_2(R/I_{Y_1}) \leq r_2(R/I_Y) < d$. Hence by Corollary 4.2, the ideal $(I_{Y_1})_{\leq d}$ is a saturated ideal defining a scheme consisting of an unmixed curve C_1 of degree s plus some zero-dimensional scheme. But C and C_1 have infinitely many hyperplane sections which agree. This means $C = C_1$ since they are unmixed. Now we claim that the ideal $\langle (I_{Y_1})_{\leq d}\rangle$ is precisely the ideal of C. We know that $\langle (I_{Y_1})_{\leq d}\rangle$ coincides with I_C in degree $\leq d$. Hence if we prove that C is d-regular then we will have proved our claim. But this follows from Proposition 2.18 since $d > r_2(R/I_C) = r_2(R/I_{Y_1})$ and (4.7) holds. This proves (c), the first half and second part of (e). Now we will prove (f). Suppose that $h^1(\mathcal{I}_{C_{\mathrm{red}}}(d-1)) = 0$. Note that we know $I_C \subset I_{C_{\mathrm{red}}} \subset I_{Y_1}$ since Y_1 is reduced, so we have that $(I_C)_t = (I_{C_{\mathrm{red}}})_t$ for all $t \leq d+1$. Hence $d > r_2(R/I_{C_{\mathrm{red}}}) = r_2(R/I_C)$. This implies $I_{C_{\mathrm{red}}}$ is d-regular (Proposition 2.18) and we get $C = C_{\mathrm{red}}$. This means the scheme V defined by the saturated ideal $\langle (I_Y)_{\leq d}\rangle$ is reduced (the proof is exactly same as that of [4]). □

Remark 4.7. Theorem 4.6 generalizes the analogous result (Corollary 3.7) in [4]: we remove the condition $d \geq s$. The next result is the main consequence for us, and introduces the key study of seeing what happens when we replace "$d \geq s$" with "$d > r_2(R/I_Y)$." It generalizes [4] Theorem 3.6. The bulk of this study will be in the next section when we consider uniform position, UPP. For now, though, we note that the only conclusion of [4] Theorem 3.6 that is not included in our Theorem 4.6 is that we do not claim that V is reduced, except in part (f), with an extra hypothesis. In the next section we will see that this is not necessarily true without that hypothesis.

Corollary 4.8. *Let $Y \subset \mathbb{P}^{n-1}$, $n \geq 3$, be a closed subscheme of dimension ≤ 1 in $\mathbb{P}^{n-1}$. Assume that*

$$\Delta H(Y, d) = \Delta H(Y, d+1) = s \neq 0$$

for some $d > r_2(R/I_Y)$. Then (a)-(f) of Theorem 4.6 continue to hold.

Proof. This follows from Corollary 4.3 and Theorem 4.6. □

5. Application to points with UPP

In this section we add the hypothesis that our finite set of points has the Uniform Position Property (UPP). The following was shown in Theorem 4.7 of [4].

Theorem 5.1. *Let $Z \subset \mathbb{P}^{r+1}$ be a reduced finite set of points with UPP. Assume that $\Delta H(R/I_Z, d) = \Delta H(R/I_Z, d+1) = s$, $d \geq s$. Then there exists a reduced, irreducible curve C of degree s such that*

(a) *$Z \subset C$;*
(b) *$I_C = \langle (I_Z)_{\leq d} \rangle$;*
(c) *$\Delta H(R/I_Z, t) = \Delta H(R/I_C, t)$ for all $t \leq d+1$.*

In this section we assume that Z has UPP, and we see what consequences this has under our hypothesis $d > r_2(R/I_Z)$. We also give several examples of "expected" behavior that does *not* occur. In doing so, we give some new insight into what can be the Hilbert function of a set of points Y with UPP. We stress again that this hypothesis differs from the analogous one in [4] only in that there it was assumed that $d \geq s$.

We begin this section by seeing what is still true for points with UPP. The rest of this section focuses on what is no longer true (even if counterexamples are not so easy to come by).

Theorem 5.2. *Let $Z \subset \mathbb{P}^{n-1}$ be a reduced finite set of points with UPP and let $(1, h_1, \ldots, h_t)$ be the h-vector of Z. Assume that*

$$h_d = h_{d+1} = s \neq 0$$

for some $d > r_2(R/I_Z)$. Then there exists an unmixed curve C of degree s such that

(a) *$Z \subset C$*
(b) *$I_C = \langle (I_Z)_{\leq d} \rangle$*
(c) *$\Delta H(C, t) = h_t$ for all $t \leq d+1$.*

Proof. By Theorem 4.6 and Corollary 4.8, we know that there exists an unmixed curve C of degree s containing a subset $Z_1 \subset Z$, such that I_{Z_1} agrees with I_C up to degree $d+1$. Since Z has UPP, $(I_Z)_t = (I_{Z_1})_t$ for $t \leq d+1$. Hence I_Z, I_{Z_1} and I_C all agree up to degree $d+1$. On the other hand, Corollary 4.8 (e) says that $\Delta H(R/I_{Z_1}, t) = \Delta H(R/I_Z, t)$ for $t \geq d$, so we get that $Z = Z_1$. This proves (a), (b) and (c). □

Apart from the hypotheses on d, the only difference between Theorem 5.1 and Theorem 5.2 is that in the former (with the stronger hypothesis $d \geq s$), it was possible to conclude that C was reduced and irreducible.

Another property that one might expect from a set of points with UPP is the following. Suppose that $\Delta H(R/I_Z, d) = \Delta H(R/I_Z, d+1) = s > \Delta H(R/I_Z, d+2)$ for some $d > r_2(R/I_Z)$. Then one might expect that the first difference $\Delta H(R/I_Z)$ is strictly decreasing from that point, i.e. $\Delta H(R/I_Z, t) > \Delta H(R/I_Z, t+1)$ for all $t \geq d+1$, until it becomes zero. Corollary 5.4 shows that in fact this is true if $d \geq s$, the assumption from [4] (although it was not proved there), and Proposition 5.3 gives a more general condition that guarantees this property.

Proposition 5.3. *Let Z be a zero-dimensional scheme with*

$$s = \Delta H(R/I_Z, d) = \Delta H(R/I_Z, d+1) > \Delta H(R/I_Z, d+2)$$

for $d > r_2(R/I_Z)$, and assume that the unmixed curve C of degree s guaranteed by Corollary 4.8 is reduced and irreducible and contains all of Z. Then

(5.1)
$$\Delta H(R/I_Z, t) > \Delta H(R/I_Z, t+1) \quad \text{for } t \geq d+1 \text{ as long as } \Delta H(R/I_Z, t) > 0.$$

In particular, suppose that Z is a reduced set of points with UPP. Then if C is reduced and irredcible, we get (5.1).

Proof. If it does not hold then say $\Delta H(R/I_Z, e) = \Delta H(R/I_Z, e+1) = s' > 0$ for some $e \geq d+2$. We already know that $s' < s$, thanks to Proposition 4.1 and the hypothesis $\Delta H(R/I_Z, d+1) > \Delta H(R/I_Z, d+2)$. We obtain a curve C' of degree s', thanks to Theorem 4.2, and clearly C' is contained in C. But since C is reduced and irreducible, this is impossible. $\square$

Corollary 5.4. *If Z is a reduced set of points with UPP, and*
$$s = \Delta H(R/I_Z, d) = \Delta H(R/I_Z, d+1) > \Delta H(R/I_Z, d+2)$$
for $d \geq s$ then (5.1) holds.

Proof. This follows from [4] Theorem 4.7, and Proposition 5.3 above. $\square$

Remark 5.5. Notice again that Theorem 5.2 does not claim that C is reduced or irreducible, even though these conclusions do appear in [4] in the analogous results, under the stronger assumption that $d \geq s$. In the examples that follow, we will see that these conclusions may in fact be false under only the assumption $d > r_2(R/I_Y)$! This is surprising since prior to this our results matched those of [4] very closely, and since UPP is strongly tied to irreducibility in many ways (see for instance the discussion in the introduction).

Example 5.6. We first recall a beautiful example from [7], which will set the stage for our first main example. Consider the family of complete intersection ideals $I_{m,n} := (x^m t - y^m z, z^{n+2} - xt^{n+1}) \subset k[x,y,z,t]$. Then for $m, n \geq 1$, $\operatorname{reg}(I_{m,n}) = m+n+2$ while $\operatorname{reg}(\sqrt{I_{m,n}}) = mn+2$. This settles (in an unexpected way) a long-standing and very interesting question (cf. [24]) of whether it is possible that the regularity increases if one replaces an ideal by its radical.

Now, we take $m = n = 4$. The complete intersection $I_{4,4}$ is then of type $(5,6)$, has degree 30, and has a Hilbert function whose first difference is

deg	0	1	2	3	4	5	6	7	8	9	10	11	12
ΔH	1	3	6	10	15	20	24	27	29	30	30	...	

On the other hand, $\sqrt{I_{4,4}}$ can be computed on a computer program, e.g. `macaulay` [1]. It has degree 26 (hence $I_{4,4}$ is not reduced), and its Hilbert function has first difference

deg	0	...	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
ΔH	1	...	24	27	29	29	29	29	28	28	28	27	27	27	26	26	...

(where the early degrees agree with $I_{4,4}$).

We focus on the ideal $\sqrt{I_{4,4}}$. Of course it defines a reduced subscheme of $\mathbb{P}^3$. One can verify on the computer (but see below as well) that $r_2(R/\sqrt{I_{4,4}}) = 8$, as one would expect. Setting $d = 10$, though, one sees that $d > r_2(\sqrt{I_{4,4}})$ and $\Delta H(R/\sqrt{I_{4,4}}), d) = \Delta H(R/\sqrt{I_{4,4}}) = 29$, so $\langle(\sqrt{I_{4,4}})_{\leq 10}\rangle$ is the saturated ideal of a curve of degree 29 (Theorem 4.6 above). One can again check that this curve is

not reduced, by checking that its radical is precisely $\sqrt{I_{4,4}}$ (which has degree 26), so one also sees that it is not irreducible (by degree considerations).

Example 5.6 does not involve a finite set of points. However, we will now give a refinement of this (Example 5.7). This example, together with Example 5.8 and Example 5.9, show some interesting and unexpected possible behavior of points with UPP. Contrast them with Theorem 5.1 above.

Example 5.7. We again work in the ring $R = k[x, y, z, t]$ and consider the ideal $I_\lambda = (z, t)$. This defines a line, λ, in $\mathbb{P}^3$. Let $F \in I_\lambda$ be a homogeneous polynomial of degree 5 defining a smooth surface containing λ (in particular, it is smooth at all points of λ). By slight abuse of notation, we will use F also for the surface. Consider the ideal $I' = I_\lambda^5 + (F)$. It can be checked from the exact sequence

$$0 \to I_\lambda^4(-5) \to I_\lambda^5 \oplus R(-5) \to I_\lambda^5 + (F) \to 0$$

(or by direct computation on the computer) that I' is the saturated ideal of a curve (not arithmetically Cohen-Macaulay) of degree 5, corresponding to the divisor $D := 5\lambda$ on the smooth surface F. Consider the linear system $|6H - D|$ on F. We first claim that this linear system has no base locus. Indeed, consider first the curve λ and the linear system $|H - \lambda|$. By considering the pencil of planes through λ and the residual cut out on F, we see that this linear system has no base locus (otherwise a point P of the base locus would be a singular point of F). Hence $|5H - D|$ has no base locus, since a union of five planes containing λ also contains D, so all the more $|6H - D|$ has no base locus.

By Bertini's theorem, then, the general element C of $|6H - D|$ is smooth. Notice that the ideal I' contains sextics in addition to the elements of $(I_\lambda^5)_6$, so the general element of $|6H - D|$ is also irreducible.

What is this curve C? It is simply the residual to D (viewed now as a curve in $\mathbb{P}^3$) in the complete intersection of F and a general element $G \in I'$ of degree 6. So this residual has degree $30 - 5 = 25$. Its Hilbert function turns out to have first difference

deg	0	...	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
ΔH	1	...	27	29	29	29	29	28	28	28	27	27	27	26	26	26	25	...

(the entries of low degree are the same as in Example 5.6; all the values after degree 21 are 25). Note that this Hilbert function agrees with that of the example of Chardin and D'Cruz (Example 5.6) up to and including degree 20. What we know that is new (and not true in their example) is that C is smooth.

Now let Z consist of sufficiently many generally chosen points on C. Z has UPP since C is smooth and irreducible, and by choosing enough points we can assume that the Hilbert function of Z agrees with that of C past degree 21. Then $r_2(R/I_Z) = 8$ and $\Delta H(R/I_Z, 10) = \Delta H(R/I_Z, 11) = 29$. The ideal generated by the components of degrees ≤ 10 then defines a curve of degree 29. Since this ideal is contained in I_Z, this curve contains C (and hence also Z). But in fact this curve of degree 29 consists of the residual in the complete intersection of degree 30 to the line λ, hence it consists of the union of C and a curve of degree 4 supported on λ. So this curve defined by the degree 10 component of I_Z is neither reduced nor irreducible, even though Z has UPP.

To summarize, this example shows that a set of points Z with UPP, for which

$$\Delta H(R/I_Z, d) = \Delta H(R/I_Z, d + 1) = s > \Delta H(R/I_Z, d + 2) \text{ for } d > r_2(R/I_Z),$$

can have the component of degree d define a non-reduced, non-irreducible curve (but it has to have degree s), and the Hilbert function can fail to be strictly decreasing past degree $d+2$.

One might wonder if Example 5.7 was somehow an "accident," in the sense that while the component of I_Z in question was neither reduced nor irreducible, still all the points of Z lay one one irreducible and reduced component. We now make a minor adjustment to show that even this is not necessarily true.

Example 5.8. In Example 5.7, instead of choosing "sufficiently many" points on C, instead choose Z to consist of 192 general points on C and one general point of λ. The first difference of the Hilbert function of Z is

deg	0	1	2	3	4	5	6	7	8	9	10	11
ΔH	1	3	6	10	15	20	24	27	29	29	29	0

Note that this is exactly the same as what we would have if we had taken 193 general points of C. Again, $r_2(R/I_Z) = 8$. The base locus of $(I_C)_9$ and $(I_C)_{10}$ is exactly the non-reduced and reducible curve of degree 29 mentioned in Example 5.7, so the Hilbert function of Z is the truncation of the Hilbert function given above. And by the general choice of the points, this will continue to be true regardless of which subsets we take. Hence Z has UPP and satisfies $\Delta H(R/I_Z, d) = \Delta H(R/I_Z, d+1) = s$ for some $d > r_2(R/I_Z)$, but not all of the points of Z lie on one reduced and irreducible component of the curve of degree s obtained by our result (since one point lies on the non-reduced component).

Again, one may wonder if it could happen that at least "almost all" of the points must lie on one reduced and irreducible component. We now give an example where the entire curve arising from Theorem 5.2 is reduced, but consists of two irreducible components, and each of these contains half of the points; but yet the points still have UPP.

Example 5.9. As remarked above, it is shown in [4] that if Γ has UPP and if $\Delta H(R/I_\Gamma, d) = \Delta H(R/I_\Gamma, d+1) = s$ for $d \geq s$ then Γ lies on a reduced and irreducible curve of degree s. We now show that with only the assumption $d > r_2(R/I_\Gamma)$, this curve need not be irreducible (although in Theorem 5.2 above we did prove the existence of this curve).

Let Q be a smooth quadric surface in $\mathbb{P}^3$, and by abuse of notation we use the same letter Q to denote the quadratic form defining Q. Let C_1 be a *general* curve on Q of type $(1, 15)$, and let C_2 be a *general* curve on Q of type $(15, 1)$. Hence both C_1 and C_2 are smooth rational curves of degree 16, and $C := C_1 \cup C_2$ is the complete intersection of Q and a form of degree 16. Note that C is arithmetically Cohen-Macaulay, but C_1 and C_2 are not.

It is not difficult to compute the Hilbert functions of these curves. We record their first differences (of course there is no difference between behavior of C_1 and behavior of C_2; this will be important in our argument):

degree	0	1	...	7	8	9	10	11	12	13	14	15	16	17	18
$\Delta H(R/I_C, -)$	1	3	...	15	17	19	21	23	25	27	29	31	32	32	32
$\Delta H(R/I_{C_i}, -)$	1	3	...	15	17	19	21	23	25	27	29	16	16	16	16

We now observe:

(1) These first differences (hence the ideals themselves) agree through degree 14, and in fact the only generator before degree 15 is Q.
(2) By adding these values, we see that $H(R/I_C, 18) = 352$ and $H(R/I_{C_i}, 15) = 241$.
(3) Since C and C_i are curves, these values represent the Hilbert functions of $I_C + (L)$ and $I_{C_i} + (L)$ for a general linear form L.
(4) $r_2(R/I_C) = 16$ since C is an arithmetically Cohen-Macaulay curve.

Let Γ_1 (respectively Γ_2) be a general set of 176 points on C_1 (respectively C_2). So $\Gamma := \Gamma_1 \cup \Gamma_2$ is a set of 352 points whose Hilbert function agrees with that of C through degree 18. In particular, we have $\Delta H(R/I_\Gamma, 17) = \Delta H(R/I_\Gamma, 18) = 32 = \deg C$. Furthermore, $r_2(R/I_\Gamma) = 16$ by (4) above. Hence Theorem 5.2 applies, and we indeed have that the component of I_Γ in degree 17 defines C. However, C is not irreducible. Our goal is to show that Γ has the Uniform Position Property, and thus there is no chance of showing that all the points must lie on a unique irreducible component in Theorem 5.2, under our hypothesis (as was done in [4]).

To show UPP, it is enough to show that the union of any choice of t_1 points of Γ_1 (i.e. t_1 general points of C_1 for $t_1 \leq 176$) and t_2 points of Γ_2 (i.e. t_2 general points of C_2 for $t_2 \leq 176$) has the truncated Hilbert function. For example, if $t_1 = 150$ and $t_2 = 160$ then we have to show that $\Delta H(R/I_\Gamma)$ has values

$$1 \quad 3 \quad 5 \quad 7 \quad 9 \quad 11 \quad 13 \quad 15 \quad 17 \quad 19 \quad 21 \quad 23 \quad 25 \quad 27 \quad 29 \quad 31 \quad 32 \quad 22 \quad 0.$$

Notice that we know that some subset has this Hilbert function, by [15]. We have to show that *all* subsets have this Hilbert function.

Let $Y = Y_1 \cup Y_2$ be our choice of t_1 points of Γ_1 and t_2 points of Γ_2. Without loss of generality, we may assume that $t_1 \geq t_2$. Hence because of the indistinguishability of C_1 and C_2 and the generality of the points, for any degree d it may be that every element of $(I_Y)_d$ vanishes on all of $C = C_1 \cup C_2$, or it may be that every element of $(I_Y)_d$ vanishes identically on C_1 but not C_2 (since $t_1 \geq t_2$), but (*) *it cannot happen that every element of $(I_Y)_d$ vanishes identically on C_2 but not on all of C_1.*

Now, we have to determine the Hilbert function of R/I_Y. *The Hilbert function of the union does not depend on what order we consider these points.* First consider Y_1. Since Y_1 consists of t_1 general points of C_1, the Hilbert function of R/I_{Y_1} is the truncation of that of R/I_{C_1}. Thanks to item (2.) above, and since $t_1 \leq 176 < 241$, the Hilbert function of R/I_{Y_1} is the truncation of R/I_C as well. We now add the points of Y_2 one by one.

Suppose that at some point, adding a point P of Y_2 does not result in a Hilbert function that is a truncation of R/I_C. Suppose that previous to P we have gone through a subset $Y_2' \subset Y_2$ and gotten a truncated Hilbert function each time, so P is the first time that this fails. This means that there is some degree d such that

$$\Delta H(R/I_C, d) > \Delta H(R/I_{Y_1 \cup Y_2'}, d) = \Delta H(R/I_{Y_1 \cup Y_2' \cup P}, d).$$

In other words, every form of degree d containing $Y_1 \cup Y_2'$ also contains P, but there is some form F of degree d containing $Y_1 \cup Y_2'$ but not all of C. Now, P is a general point of C_2 independent of the choices of points previous to it, and every element of $(I_{Y_1 \cup Y_2'})_d$ vanishes at P. Therefore every element of $(I_{Y_1 \cup Y_2'})_d$ vanishes on an open subset of C_2, and hence (by irreducibility) on all of C_2. Since there is an $F \in (I_{Y_1 \cup Y_2'})_d$ not vanishing on all of C, F must not vanish on all of C_1. But our

conclusion (*) above used only the fact that $t_2 \leq t_1$, so it remains true replacing Y_2 by $Y_2' \cup P$. Contradiction. Therefore Γ has UPP.

6. The connection to the Weak Lefschetz Property

In the previous sections, we gave the results on the Hilbert function of the scheme Y of $\dim(Y) \leq 1$ in degree $d > r_2(R/I_Y)$ in terms of its first difference. The range in which we obtained our results was $d > r_2(R/I_Y)$. We also gave examples to show that we cannot expect these results to extend to the range $d \leq r_2(R/I_Y)$ (Remark 4.5). However, in this section we show that in fact there are useful analogous results that we can obtain in this smaller range, but changing the focus somewhat (Proposition 6.1 through Theorem 6.4). As an application, we give new results in terms of second difference of the Hilbert function if Y is a set of points with WLP in $\mathbb{P}^{n-1}$, $n > 3$. In particular, we apply these result to a set of points with UPP and WLP in $\mathbb{P}^3$ (Theorem 6.8 and Corollary 6.12). It is a natural question whether every set of points with UPP has WLP, since both are open properties. (It is easy to find examples having WLP but not UPP.) We give an example (Example 6.9) to show that this is not the case.

First we prove analogies to Proposition 4.1 and Theorem 4.2 for $d \leq r_2(R/I_Y)$.

Proposition 6.1. *Let Y be a scheme of dimension ≤ 1 in $\mathbb{P}^{n-1}$, $n > 3$. Let $K = (I_Y + (L_1, L_2))/(L_1, L_2)$; ;this is a homogeneous ideal in $S = R/(L_1, L_2)$. Then*

$$H(S/K, d) \geq H(S/K, d+1) \tag{6.1}$$

for $d \geq r_3(R/I_Y)$. Moreover, for $r_2(R/I_Y) > d > r_3(R/I_Y)$, we have equality in (6.1) *if and only if* $\mathrm{Gin}(K)$ *has no minimal generators in degree $d+1$.*

Proof. The proof can be given in the same manner as that of Proposition 4.1 (though not trivially). □

Proposition 6.2. *Under the same situation as Proposition 6.1, if we have equality in* (6.1)*, say s, for $r_2(R/I_Y) > d > r_3(R/I_Y)$ then $\langle (I_Y)_{\leq d}\rangle^{\mathrm{sat}}$ is a homogeneous ideal defining a two dimensional subscheme of degree s in $\mathbb{P}^{n-1}$.*

Proof. If we consider the homogeneous ideal of $S = R/(L_1, L_2)$

$$M = \frac{(\langle (I_Y)_{\leq d}\rangle + (L_1, L_2))}{(L_1, L_2)}$$

then M is d-regular by Theorem 2.13 and Proposition 6.1. Consider the generic initial ideal of M. Since M is d-regular for $r_3(R/I_Y) < d < r_2(R/I_Y)$, M could not have any power of x_{n-2}. This means that $D(M) = n-3$, since $d > r_3(R/I_Y)$ implies that the degree d component of $Gin(I_Y)$ contains the monomial x_{n-3}^d. Note that $\mathrm{Gin}(M) = (\mathrm{Gin}(\langle (I_Y)_{\leq d}\rangle)_{x_n \to 0})_{x_{n-1}\to 0}$ does not have any minimal generators of degree larger than d. Hence every minimal generator of $\langle (I_Y)_{\leq d}\rangle$ in degree $> d$ must involve the variables x_{n-1} or x_n. This implies $D(\langle (I_Y)_{\leq d}\rangle) = n-3$. Hence $\langle (I_Y)_{\leq d}\rangle^{\mathrm{sat}}$ defines a two dimensional subscheme of $\mathbb{P}^{n-1}$.

It follows from $\Delta^2 H(R/\langle (I_Y)_{\leq d}\rangle, t) = H(S/M, t) = s$ for sufficiently large t that the degree of $(R/\langle (I_Y)_{\leq d}\rangle^{\mathrm{sat}})$ is exactly s. □

Corollary 6.3. *Let Y be a scheme of dimension ≤ 1 in $\mathbb{P}^3$. If we have equality in* (6.1)*, say s, for $r_2(R/I_Y) > d > r_3(R/I_Y)$, then $(I_Y)_d$ has a common factor F of degree s.*

Proof. Let X be the unmixed part of $\langle (I_Y)_{\leq d}\rangle$ of $\dim(X) = 2$. Then $I_X = (F)$ for a homogeneous polynomial F since X has codimension 1 in $\mathbb{P}^3$. Hence $(I_Y)_d \subset (I_X)_d \subset (F)_d$ and thus $(I_Y)_d$ has a common factor F of degree s. □

We need an additional condition to be able to deduce the saturatedness and regularity of $\langle (I_Y)_{\leq d}\rangle$.

Theorem 6.4. *Let Y be a scheme of dimension ≤ 1 in $\mathbb{P}^{n-1}$, $n > 3$. Let $K = (I_Y + (L_1, L_2))/(L_1, L_2) \subset S = R/(L_1, L_2)$. Suppose that $H(S/K, d) = H(S/K, d+1)$ for $r_2(R/I_Y) > d > r_3(R/I_Y)$. Then $\langle (I_Y)_{\leq d}\rangle$ is a homogeneous ideal (not necessarily saturated) defining a two dimensional scheme of degree s in $\mathbb{P}^{n-1}$. Furthermore, the following are equivalent.*

(a) *$\langle (I_Y)_{\leq d}\rangle$ is d-regular.*

(b) *$0 \to (R/I_Y + (L_1))_d \xrightarrow{\times L_2} (R/I_Y + (L_1))_{d+1}$ for a general linear form L_2.*

If these equivalent conditions are satisfied then the ideal is saturated as well.

Proof. The first assertion is just a repetition of Proposition 6.2, just for completeness. We now prove the equivalence. Let $\bar{I} = \langle (I_Y)_{\leq d}\rangle$. For a general linear form L_2 if the map

$$(R/I_Y + (L_1))_d \xrightarrow{\times L_2} (R/I_Y + (L_1))_{d+1}$$

is not injective, then we know that there is a minimal generator of $\mathrm{Gin}(\bar{I})$ with degree $d+1$ from the proof of Proposition 4.1. But this means $\langle (I_Y)_{\leq d}\rangle$ is not d-regular (Theorem 2.12).

Conversely, suppose that the map

$$(R/I_Y + (L_1))_d \xrightarrow{\times L_2} (R/I_Y + (L_1))_{d+1}$$

is injective for a general linear form L_2. It is enough to show that $\mathrm{Gin}(\bar{I})$ has no minimal generator of degree $d+1$ in order to prove $\langle (I_Y)_{\leq d}\rangle$ is d-regular. Note that $(\mathrm{Gin}(\bar{I})_{x_n\to 0})_{x_{n-1}\to 0}$ is d-regular. Hence if $\mathrm{Gin}(\bar{I})_{x_n\to 0}$ has a minimal generator of degree $d+1$ then it must involve the variable x_{n-1}. But this is impossible because the map $(R/I_Y+(L_1))_d \longrightarrow (R/I_Y+(L_1))_{d+1}$ is injective. Hence $\mathrm{Gin}(\bar{I})_{x_n\to 0}$ is d-regular (Theorem 2.13). Suppose that there is a minimal generator x^L of $\mathrm{Gin}(\bar{I})$ with degree $d+1$. Then it must involve the variable x_n. Notice that $\mathrm{Gin}(\bar{I})_{d+1} \subset \mathrm{Gin}(I_Y)_{d+1}$ and x^L cannot be a minimal generator of $\mathrm{Gin}(I_Y)$ since I_Y is a saturated ideal. Hence we can choose a monomial $x^T \in \mathrm{Gin}(I_Y)_d$ such that $x_i x^T = x^L$ for some variable x_i. But it contradicts the fact that x^L is a minimal generator of $\mathrm{Gin}(\bar{I})$ because $x^T \in \mathrm{Gin}(\bar{I})_d = \mathrm{Gin}(I_Y)_d$. Hence $\bar{I}$ is d-regular (Theorem 2.13).

The proof that then $\bar{I}$ is saturated is exactly same as that of Theorem 4.2 □

Remark 6.5. The proof of Theorem 6.4 makes it clear that if

$$(R/I_Y + (L_1))_d \xrightarrow{\times L_2} (R/I_Y + (L_1))_{d+1}$$

is not injective then $(I_Y)_{\leq d}$ is not d-regular. However, the proof leaves it unclear whether $(I_Y)_{\leq d}$ is saturated in this case. In the following example, we show that this ideal may or may not be saturated. We used `macaulay` for this calculation, but it could also be computed by hand.

Example 6.6. Let $J = (F_1, F_2, F_3) \subset R$ be a regular sequence of type $(2, 2, 2)$ defining a zero-dimensional complete intersection Z_1 of degree 8 in $\mathbb{P}^3$, and let Q be a general quadratic polynomial. (It is enough that Q not vanish on any of the 8 points of Z. We will also use Q to denote the corresponding quadric surface.) Let $I = (QF_1, QF_2, QF_3)$. This is the saturated ideal of the union of Z_1 and Q. Let Z_2 be a set of 81 general points on Q, and let $Y = Z_1 \cup Z_2$. As above, let L_1 and L_2 be general linear forms. The Hilbert functions of the relevant ideals are as follows.

degree	0	1	2	3	4	5	6	7	8	9	...
$H(R/I_Y, -)$	1	4	10	20	32	44	57	72	89	89	...
$H(R/(I_Y + (L_1)), -)$	1	3	6	10	12	12	13	15	17	0	...
$H(R/(I_Y + (L_1, L_2)), -)$	1	2	3	4	2	2	2	2	2	0	...

From this we see that $r_2(R/I_Y) = 8$, and it is not hard to see that $r_3(R/I_Y) = 3$.

Let $d = 4$. We have the needed equality in Corollary 6.3, so $(I_Y)_4$ has a common factor of degree 2, namely Q. In fact, $(I_Y)_{\leq 4}$ is exactly I, and it is not hard to check that its regularity is 6 (since the regularity of (F_1, F_2, F_3) is 4). Since $H(R/(I_Y + (L_1, L_2)), 5) = 2$, we see that we do not have the injectivity of the desired map. Hence we confirm in this example what is proved in Theorem 6.4, that we do not have regularity in degree $d = 4$. However, as mentioned above, $(I_Y)_{\leq 4} = I$ is saturated. Note that the non-injectivity implies that Y does not have the Weak Lefschetz Property.

Now we modify this example a little bit. Let Z_1 now consist of 16 general points in $\mathbb{P}^3$ and let Q again be a quadric surface not containing any of the points of Z_1. Note that I_{Z_1} has four cubic generators and three quartic generators, but the component in degree 3 does define Z_1 scheme-theoretically. Let Z_2 again consist of 81 general points of Q, and let $Y = Z_1 \cup Z_2$. We have the following Hilbert functions.

degree	0	1	2	3	4	5	6	7	8	9	...
$H(R/I_Y, -)$	1	4	10	20	35	52	65	80	97	97	...
$H(R/(I_Y + (L_1)), -)$	1	3	6	10	15	17	13	15	17	0	...
$H(R/(I_Y + (L_1, L_2)), -)$	1	2	3	4	5	2	2	2	2	0	...

This time $r_2(R/I_Y)$ is again 8, but $r_3(R/I_Y) = 4$. Let $d = 5$. We again have the needed equality, so $(I_Y)_5$ has a common factor of degree 2, namely Q.

The map

$$(R/I_Y + (L_1))_5 \xrightarrow{\times L_2} (R/I_Y + (L_1))_6$$

obviously has no chance to be injective. Hence by Theorem 6.4, $(I_Y)_{\leq 5}$ is not 5-regular. What about saturation? Notice that $(I_Y)_{\leq 5}$ is just $Q \cdot (I_{Z_1})_3$, so this defines $Q \cup Z_1$ scheme-theoretically but is not saturated. Notice that if we instead take $d = 6$, the ideal is both 6-regular and saturated, and the corresponding map is injective.

Theorem 6.7. *Let Z be a zero-dimensional subscheme of $\mathbb{P}^{n-1}$, $n > 3$, with* WLP. *Suppose that*

$$\Delta^2 H(R/I_Z, d) = \Delta^2 H(R/I_Z, d+1) = s$$

for $r_2(R/I_Z) > d > r_3(R/I_Z)$, where $\Delta^2 H(R/I_Z, \cdot)$ is the second difference of Hilbert function of R/I_Z. Then $\langle (I_Z)_{\leq d} \rangle$ is a saturated ideal defining a two dimensional subscheme of degree s in $\mathbb{P}^{n-1}$ and it is d-regular.

Proof. With notation of Proposition 4.1, consider the exact sequence (4.3). By definition of WLP, we know that the map

$$(R/I_Z + (L_1))_d \xrightarrow{\times L_2} (R/I_Z + (L_1))_{d+1}$$

is injective $d < r_2(R/I_Z)$ and

$$\Delta^2 H(R/I_Z, t) = H(R/(I_Z + J), t)$$

for $t \le r_2(R/I_Z)$ where $J = (L_1, L_2)$ for general linear forms L_1 and L_2. Then the result follows from Theorem 6.4. □

Theorem 6.8. *Let Z be a set of points with UPP in $\mathbb{P}^3$. Let $I_Z = (F_1, \dots, F_m)$ be the defining ideal of Z, where $\deg(F_i) = d_i$ and $d_1 \le d_2 \le \cdots \le d_m$. Let $K = (I_Z + (L_1, L_2))/(L_1, L_2)$ and let $S = R/(L_1, L_2)$. Then*

$$H(S/K, d) > H(S/K, d+1) \tag{6.2}$$

for $d_2 \le d \le r_2(R/I_Z)$ and $H(S/K, d) = 0$ for $d > r_2(R/I_Z)$.

Proof. First note that if $d \ge r_2(R/I_Z)$ then the assertions are just from the definition of $r_2(R/I_Z)$. If $H(S/K, d) = H(S/K, d+1)$ for $d < r_2(R/I_Z)$ then we know that $(I_Z)_d$ has a common factor F by Corollary 6.3. Then, F must be irreducible and $(I_Z)_{\le d} = (F)$ since Z has UPP (Lemma 4.4 in [4]). This means that $d_1 \le d < d_2$ and $\deg(F) = d_1$. □

Example 6.9. In Remark 2.16 we discussed the connection between the calculation of the second reduction number for a zero-dimensional scheme and the Weak Lefschetz Property (WLP). WLP should be viewed as being a "general" property, say for a fixed Hilbert function (as long as the Hilbert function does not already exclude the property). But UPP is also a "general" property. It is easy to see that a set of points can have WLP and not UPP (see [18] for information on the Hilbert function of a set of points with WLP). In this example we exhibit a set of points, Γ, that has UPP but not WLP.

Let C be a smooth arithmetically Buchsbaum curve in $\mathbb{P}^3$ whose deficiency module $M(C)$ is one-dimensional in each of two consecutive degrees, and zero elsewhere. The Buchsbaum property means that all linear forms annihilate $M(C)$. The existence of a smooth curve in any even liaison class of curves in $\mathbb{P}^3$ is proved in [23]. The precise shifts of $M(C)$ where smooth curves exist were computed in [5] in a more general setting. In this case, we may assume that $M(C)$ is non-zero in degrees 3 and 4.

Let L be a general linear form, defining a plane $H \subset \mathbb{P}^3$. Let I_C be the saturated ideal of C. We have

$$J := I_C/(L \cdot I_C) \cong \frac{I_C + (L)}{(L)} \subset R/(L) := S$$

and a short exact sequence of sheaves

$$0 \to \mathcal{I}_C(-1) \xrightarrow{\times L} \mathcal{I}_C \to \mathcal{I}_{C\cap H, H} \to 0.$$

Taking cohomology on this last sequence and combining it with the isomorphisms above, we obtain

$$\frac{I_{C\cap H,H}}{J} \cong M(C)(-1) \tag{6.3}$$

since C is arithmetically Buchsbaum, where $I_{C\cap H,H}$ is the saturation of $\mathcal{I}_{C\cap H,H}$.

Now let Γ consist of a general set of sufficiently many points on C. The set Γ has the Uniform Position Property since it consists of a general set of points on a smooth curve. We want to show that Γ does not have the Weak Lefschetz Property.

We have $(I_C)_t = (I_\Gamma)_t$ for all $t \le t_0$ for some t_0, which we may make as large as necessary by choosing sufficiently many points for Γ. Since the points are general, L does not vanish on any of them. Then we have a short exact sequence of sheaves

$$0 \to \mathcal{I}_\Gamma(-1) \to \mathcal{I}_\Gamma \to \mathcal{O}_H \to 0.$$

Let $J' = I_\Gamma/(L \cdot I_\Gamma)$. We obtain the isomorphism

$$(S/J)_t = (S/J')_t \cong (\ker[H^1_*(\mathcal{I}_\Gamma)(-1) \xrightarrow{\times L} H^1_*(\mathcal{I}_\Gamma)])_t \tag{6.4}$$

for all $t \le t_0$, where H^1_* refers to a direct sum over all twists. This isomorphism commutes with multiplication by linear forms over this range of t.

Now consider the short exact sequence of sheaves

$$0 \to \mathcal{I}_C \to \mathcal{I}_\Gamma \to \mathcal{I}_{\Gamma|C} \to 0.$$

Since $H^0(\mathcal{I}_{\Gamma|C}(t)) \cong H^0(\mathcal{O}_C(tH - \Gamma))$ measures the space of hypersurface sections, up to linear equivalence, of degree t vanishing on Γ (but not on all of C), we may assume that $h^0(\mathcal{I}_{\Gamma|C}(t)) = 0$ for $t \le t_0$ (possibly changing t_0 slightly), again by taking sufficiently many points. (In principle $h^0(\mathcal{I}_{\Gamma|C}(t))$ might be non-zero in degrees 3 and 4, coming from $M(C)$, but then adding a hyperplane annihilates such a section, modulo I_C, which is clearly nonsense.) Hence we have

$$0 \to H^1(\mathcal{I}_C(t)) \to H^1(\mathcal{I}_\Gamma(t))$$

for $t \le t_0$. Therefore $M(C)$ is isomorphic to a submodule of $H^1_*(\mathcal{I}_\Gamma)$. Since $M(C)$ is annihilated by all linear forms, we may invoke (6.4) to conclude that $M(C)$ is isomorphic to a submodule of S/J'. In particular, since $M(C)$ occurs in degrees 3 and 4,

(6.5) $M(C)$ is isomorphic to a submodule of S/J' occurring in degrees 4 and 5,

thanks to the shift in (6.4).

Let L' be another general linear form, and consider the homomorphism

$$(S/J')_4 \xrightarrow{\times L'} (S/J')_5.$$

We want to show that this is neither injective nor surjective. The fact that it is not injective follows from (6.5). For the non-surjectivity, consider the commutative

diagram

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \to & \left(\dfrac{I_{C\cap H,H}}{J}\right)_4 & \to & \left(\dfrac{S}{J}\right)_4 & \to & \left(\dfrac{S}{I_{C\cap H,H}}\right)_4 & \to & 0 \\
 & & \downarrow \times 0 & & \downarrow \phi & & \downarrow & & \\
0 & \to & \left(\dfrac{I_{C\cap H,H}}{J}\right)_5 & \to & \left(\dfrac{S}{J}\right)_5 & \to & \left(\dfrac{S}{I_{C\cap H,H}}\right)_5 & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & A & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

The vertical arrows correspond to multiplication by L'. Since $S/J = S/J'$ in degrees 4 and 5 (and beyond), the middle vertical map is the one that we have to show is not surjective. The fact that the leftmost vertical map is zero comes from (6.3) and the Buchsbaum property. Thanks to (6.5), $A \neq 0$. The fact that the rightmost vertical map is injective comes from the fact that $I_{C\cap H,H}$ is a saturated ideal. Therefore we get the desired non-surjectivity of the middle column from the Snake Lemma.

Example 6.10. Uwe Nagel asked us whether our methods might be able to settle a question related to that of Example 6.9. That is, suppose that C is a smooth curve over a field of characteristic zero. Then it is well known that the general hyperplane or hypersurface section of C has UPP. Does either of these necessarily also have WLP? When $C \subset \mathbb{P}^3$, it is well known that the general hyperplane section does have WLP, since all zero-dimensional schemes in $\mathbb{P}^2$ have WLP [18].

A re-interpretation of Example 6.9 gives the surprising (to us) answer "no" to both questions. Let C be the smooth curve of degree 15 from that example, and let S be the cone over C from a general point P in $\mathbb{P}^4$. After a change of variables, the generators of I_C, viewed as polynomials in $k[x_0,x_1,x_2,x_3,x_4]$, give the generators of I_S, and S is smooth away from P. In particular, if $L = x_4$ is the linear form defining $\mathbb{P}^3$ in $\mathbb{P}^4$ (which holds without loss of generality after our change of variables), then

$$I_C = [I_S + (L)]/(L) \tag{6.6}$$

where I_C is the saturated ideal of C. It follows from a standard exact sequence that $H^1(\mathcal{I}_S(t)) = 0$ for all t, since (6.6) implies that multiplication by a general linear form induces an injection between any pair of consecutive components of $H^1_*(\mathcal{I}_S)$. Consequently, for any homogeneous polynomial F not vanishing on S (which is irreducible), $I_S + (F)$ is the saturated ideal of the hypersurface section of S by F (cf. [20] Remark 2.1.3).

Now observe that in Example 6.9, we could replace Γ by a general hypersurface section of sufficiently large degree. Indeed, what is important is that $(I_C)_t = (I_\Gamma)_t$ for all $t \le t_0$, as describe above. This immediately shows that the general

hypersurface section of a smooth curve even in $\mathbb{P}^3$ does not necessarily have WLP. But furthermore, let $L = x_4$ be as above, and let F be a general form of sufficiently large degree in $k[x_0, x_1, x_2, x_3, x_4]$, with $\bar{F}$ its restriction to $k[x_0, x_1, x_2, x_3]$. We have just observed that the zero-dimensional scheme defined by $(I_C + \bar{F})^{sat}$ fails to have WLP. But

$$[I_C + (\bar{F})]^{sat} = \left[\frac{[I_S + (L)]}{(L)} + (\bar{F})\right]^{sat} = \left[\frac{[I_S + (F)] + (L)}{(L)}\right]^{sat}.$$

Since F avoids P for a general choice of F, $I_S + (F)$ is the saturated ideal of a smooth curve, and the above equation shows that its hyperplane section is the example we have already considered, which does not have WLP.

Remark 6.11. The key idea in the preceding two examples is to use the structure of the deficiency module to force the failure of WLP. Consequently we do not know if it is true that the general hyperplane or hypersurface section of a smooth arithmetically Cohen-Macaulay curve necessarily has WLP.

Our last result gives further information about the growth of the Hilbert function if we know UPP and WLP.

Corollary 6.12. *Let Z be a set of points with UPP and WLP in $\mathbb{P}^3$. Let $I_Z = (F_1, \ldots, F_m)$ be the defining ideal of Z, where $\deg(F_i) = d_i$ and $d_1 \le d_2 \le \cdots \le d_m$. Then*

$$\Delta^2 H(R/I_Z, d) > \Delta^2 H(R/I_Z, d+1) \tag{6.7}$$

for $d_2 \le d < r_2(R/I_Z)$.

Proof. The proof follows from Theorem 6.8 since

$$\Delta^2 H(R/I_Z, t) = H(R/(I_Z + J), t)$$

for $t \le r_2(R/I_Z)$ where $J = (L_1, L_2)$ for general linear forms L_1 and L_2 by WLP. □

Remark 6.13. The result of Corollary 6.12 holds only for points in $\mathbb{P}^3$. For $\mathbb{P}^n$, $n \ge 4$, we cannot say anything about the strictly decreasing property of the second difference of the Hilbert function. However, it may well be that higher differences of the Hilbert function can be controlled in higher projective spaces.

References

[1] D. Bayer and M. Stillman, Macaulay: A system for computation in algebraic geometry and commutative algebra. Source and object code available for Unix and Macintosh computers. Contact the authors, or download from ftp://math.harvard.edu via anonymous ftp.

[2] D. Bayer and M. Stillman, *A criterion for detecting m-regularity*, Invent. Math. **87** (1987), 1-11.

[3] A. Bigatti, *Upper bounds for the Betti numbers of a given Hilbert function*, Comm. Algebra 21 (1993), no. 7, 2317-2334.

[4] A. Bigatti, A.V. Geramita and J. Migliore, *Geometric consequences of extremal behavior in a theorem of Macaulay.* Trans. Amer. Math. Soc. 346 (1994), no. 1, 203-235.

[5] G. Bolondi and J. Migliore, *Buchsbaum Liaison Classes*, J. Alg. **123** (1989), 426–456.

[6] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Stud. Adv. Math. **39**, Cambridge Univ. Press, Cambridge, U.K., 1993; revised paperback edition, 1998.

[7] M. Chardin and C. D'Cruz, *Castelnuovo-Mumford regularity: Examples of curves and surfaces*, J. Algebra **270** (2003), no. 1, 347–360.

[8] A. Conca, *Koszul homology and extremal properties of Gin and Lex*, to appear in Trans. Amer. Math. Soc.
[9] A. Conca, *Reduction numbers and initial ideals*, Proc. Amer. Math. Soc. **131** (2003), no. 4, 1015–1020
[10] D. Eisenbud, "Commutative algebra with a view toward algebraic geometry," Graduate Texts in Mathematics, 150. Springer-Verag, New York, 1995.
[11] D. Eisenbud, "The Geometry of Syzygies, A second course in Commutative Algebra and Algebraic Geometry," University of California, Berkeley, http://www.msri.org/people/staff/de/ready.pdf
[12] D. Eisenbud and S. Goto. *Linear free resolutions and minimal multiplicity*, J. Algebra **88**, (1984), 89-133.
[13] S. Eliahou and M. Kervaire, *Minimal resolutions of some monomial ideals*, J. Algebra **129**, (1990), 1-25.
[14] A. Galligo, *A propos du théorème de préparation de Weierstrrass*, in: Fonctions de Plusieurs Variables Complexes, Lecture Note in Mathematics, Springer, Berlin, 1974, pp. 543–579.
[15] A.V. Geramita, P. Maroscia and L. Roberts, *The Hilbert Function of a Reduced k-Algebra*, J. London Math. Soc. **28** (1983), 443–452.
[16] M. Green, *Generic Initial Ideals*, in Six lectures on Commutative Algebra, (Elias J., Giral J.M., Miró-Roig, R.M., Zarzuela S., eds.), Progress in Mathematics **166**, Birkhäuser, 1998, 119–186.
[17] M. Green and M. Stillman, *A tutorial on generic initial ideals*, Gröbner bases and applications (Linz, 1998), 90-108, London Math. Soc. Lecture Note Ser., **251**, Cambridge Univ. Press, Cambridge, 1998.
[18] T. Harima, J. Migliore, U. Nagel and J. Watanabe, *The Weak and Strong Lefschetz Properties for Artinian K-Algebras*, J. Algebra **262** (2003), 99-126.
[19] L.T. Hoa and N.V. Trung, *Borel-fixed ideals and reduction number*, J. Algebra **270** (2003), no. 1, 335–346.
[20] J. Migliore, "Introduction to Liaison Theory and Deficiency Modules," Birkhäuser, Progress in Mathematics 165, 1998; 224 pp. Hardcover, ISBN 0-8176-4027-4.
[21] J. Migliore and R. Miró-Roig, *Ideals of general forms and the ubiquity of the Weak Lefschetz Property*, J. Pure Appl. Algebra **182** (2003), 79–107.
[22] U. Nagel, *Castelnuovo-Regularität und Hilbertreihen*, Math. Nachr. **142** (1989), 27–43.
[23] P. Rao, *Liaison among Curves in* $\mathbb{P}^3$, Invent. Math. **50** (1979), 205–217.
[24] M.S. Ravi, *Regularity of ideals and their radicals*, Manuscripta Math. **68** (1990), 77–87.